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Predicate invalidity, §8.5

I. **Recall how we proved an argument invalid in propositional logic**
1. A ⊃ B
2. ~(B · A) /A ≡ B

Line up the premises and conclusion
A ⊃ B / ~(B · A) // A ≡ B

Assign truth values to the component sentences to form a counterexample which makes the premises true and the conclusion false

A	⊃	B	/	~	(B	·	A)	//	A	≡	B
F	**T**	T		**T**	T	F	F		F	**F**	T

So, the argument is shown invalid when A is false and B is true.

II. **Proving an argument invalid in Predicate Logic**
In Predicate Logic, we may use the method of finite universes.
If an argument is valid, then it is true, no matter what the interpretation of the constants, and no matter what exists in the world.
Logical truths are true in all possible universes.
They have no prejudices about what exists in the world.
If they did, then they wouldn't be logical truths, but empirical truths!
Even if the world has only one member, or two or three, valid arguments should be valid.

Consider the following invalid argument:
(x)(Wx ⊃ Hx)
(x)(Ex ⊃ Hx) / (x)(Wx ⊃ Ex)

Imagine there were only one object in the universe. Let's call it 'a'.
(x)(Wx ⊃ Hx) would be equivalent to Wa ⊃ Ha
(x)(Ex ⊃ Hx) would be equivalent to Ea ⊃ Ha
∴(x)(Wx ⊃ Ex) would be equivalent to Wa ⊃ Ea

Now, assign truth values, as in the propositional case to make the premises true and the conclusion false:

Wa	⊃	Ha	/	Ea	⊃	Ha	//	Wa	⊃	Ea
T	**T**	T		F	**T**	T		T	**F**	F

So, the argument is shown invalid in a one-member universe, where Wa is true, Ha is true, and Ea is false.

Not all invalid arguments are invalid in a one-member universe.
All we need is one universe in which they are invalid, to show them invalid.
Even if an argument works in a one-member universe, it might still be invalid!

III. **Universes of more than one member**

Consider the following invalid argument:

(x)(Wx ⊃ Hx)

(∃x)(Ex · Hx)   / (x)(Wx ⊃ Ex)

In a one-object universe, we have:

Wa	⊃	Ha	/	Ea	·	Ha	//	Wa	⊃	Ea
				F				T	**F**	F

There is no way to construct a counterexample.

But the argument is invalid.

We have to consider a larger universe.

If there are two objects in a universe, a and b:

(x)$\mathscr{F}$x	becomes	$\mathscr{F}$a · $\mathscr{F}$b	because every object has $\mathscr{F}$
(∃x)$\mathscr{F}$x	becomes	$\mathscr{F}$a ∨ $\mathscr{F}$b	because only some objects have $\mathscr{F}$

If there are three objects in a universe, then

(x)$\mathscr{F}$x	becomes	$\mathscr{F}$a · $\mathscr{F}$b · $\mathscr{F}$c
(∃x)$\mathscr{F}$x	becomes	$\mathscr{F}$a ∨ $\mathscr{F}$b ∨ $\mathscr{F}$c

Returning to the problem...

In a universe of two members, we represent the argument is equivalent to:

(Wa ⊃ Ha) · (Wb ⊃ Hb) /   (Ea · Ha) ∨ (Eb · Hb)   //   (Wa ⊃ Ea) · (Wb ⊃ Eb)

Now, assign values to each of the terms to construct a counterexample.

(Wa	⊃	Ha)	·	(Wb	⊃	Hb)	/	(Ea	·	Ha)	∨	(Eb	·	Hb)
T	T	T	**T**	F	T	T		F	F	T	**T**	T	T	T

//	(Wa	⊃	Ea)	·	(Wb	⊃	Eb)
	T	F	F	**F**	F	T	T

The argument is shown invalid in a two-member universe, when

Wa: true	Wb: false
Ha: true	Hb: true
Ea: false	Eb: true

IV. **Constants**

Constants get rendered as themselves. Don't expand when moving to a larger universe.

Consider:

(∃x)(Ax · Bx)

Ac   /Bc

We can't show it invalid in a one-member universe.

Ac	·	Bc	/	Ac	//	Bc
	F	F				F

We must move to a two-member universe.

Here, we generate a counter-example.

(Ac	·	Bc)	∨	(Aa	·	Ba)	/	**Ac**	//	**Bc**
T	F	F	**T**	T	T	T		**T**		**F**

This argument is shown invalid in a two-member universe, when
Ac: true
Bc: false
Aa: true
Ba: true

Some arguments need three, four, or even infinite models to be shown false.

V. **Propositions whose main connective is not a quantifier**

Consider the following argument:
(∃x)(Px · Qx)
(x)Px ⊃ (∃x)Rx
(x)(Rx ⊃ Qx)   /(x)Qx

In a 1-member universe, this gets rendered as:
Pa · Qa / Pa ⊃ Ra / Ra ⊃ Qa // Qa
But there is no counter-example.

Pa	·	Qa	/	Pa	⊃	Ra	/	Ra	⊃	Qa	//	**Qa**
	F	F										**F**

In a two-member universe, note what happens to the second premise:
(Pa · Qa) ∨ (Pb · Qb) / (Pa · Pb) ⊃ (Ra ∨ Rb) / (Ra ⊃ Qa) · (Rb ⊃ Qb) // Qa · Qb
Each quantifier is unpacked independently.
The main connective, the conditional, remains the main connective.
We can clearly see here the difference between instantiation and translation into a finite universe.

We can construct a counterexample for this argument in a two-member universe:

(Pa	·	Qa)	∨	(Pb	·	Qb)	/	(Pa	·	Pb)	⊃	(Ra	∨	Rb)
	F	F	**T**	T	T	T				T	**T**	F	T	T

/	(Ra	⊃	Qa)	·	(Rb	⊃	Qb)	//	Qa	·	Qb
	F	T	F	**T**	T	T	T		F	F	T

This argument is shown invalid in a two-member universe, when
Pa: either true or false   Pb: true
Qa: false   Qb: true
Ra: false   Rb: true

(There is another solution. Can you construct it?)

III. **Exercises**. Show each of the following arguments invalid by generating a counter-example using the method of finite universes.

1)
1. $(x)(Ex \supset Fx)$
2. $(\exists x)(Gx \cdot \sim Fx)$ / $(\exists x)(Ex \cdot \sim Gx)$

2)
1. $(x)(Bx \supset \sim Dx)$
2. $\sim Bj$ / Dj

3)
1. $(x)(Hx \supset \sim Ix)$
2. $(\exists x)(Jx \cdot \sim Ix)$ / $(x)(Hx \supset Jx)$

4)
1. $(x)(Kx \supset \sim Lx)$
2. $(\exists x)(Mx \cdot Lx)$ / $(x)(Kx \supset \sim Mx)$

5)
1. $(\exists x)(Px \cdot Qx)$
2. $(x)(Qx \supset \sim Rx)$
3. Pa / $(x)\sim Rx$

6)
1. $(x)(Ax \supset Bx)$
2. $(\exists x)(Dx \cdot Bx)$
3. $(\exists x)(Dx \cdot \sim Bx)$ / $(x)(Ax \supset Dx)$

IV. **Solutions**

Note that alternate solutions are possible.

1) Invalid in a one-member universe, where Ga: true; Ea: false; Fa: false

2) Invalid in a one-member universe, where Bj: false; Dj: false

3) Invalid in a two-member universe, where Ha: true; Ia: false; Ja: false; Hb: true or false; Ib: false; Jb: true

4) Invalid in a two-member universe, where Ka: false; La: true; Ma: true; Kb: true; Lb: false; Mb: true.

5) Invalid in a two-member universe, where Pa: true; Qa: false; Ra: true; Pb: true; Qb: true; Rb: false

6) Invalid in a three-member universe, where Aa: true; Ba: true; Da: false; Ab: true or false; Bb: true; Db: true; Ac: false; Bc: false; Dc: true