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Conditional proof, §7.5

I. **Conditional Proof: A New Method of Derivation**

When you want to derive a conditional conclusion, you can assume the antecedent of the conditional, for the purposes of the derivation, taking care to indicate the presence of that assumption later.

Procedure:
1) Indent, assuming the antecedent of your desired conditional.
Write ‘ACP’, for ‘assumption for conditional proof’.
Use a vertical line to set off the assumption from the rest of your derivation.
2) Derive the consequent of desired conditional.
Continue the vertical line.
Proceed otherwise as before, using any lines already established.
3) Discharge (un-indent).
Write the first line of your assumption, a horseshoe, and the last line of the indented sequence.
Justify the un-indented line with CP, and indicate the indented line numbers.

So, consider:

1. $A \vee B$		
2. $B \supset (E \cdot D)$	/ $\sim A \supset D$	Note the conditional conclusion.
	3. $\sim A$	ACP	What if $\sim A$ were true (i.e. A were false)?
	4. B	1, 3, DS	
	5. $E \cdot D$	2, 4, MP	
	6. D	5, Com, Simp	Then D would be true.
7. $\sim A \supset D$	3-6, CP	So, if A were true, then D would be.

QED

Note that once you’ve discharged, you may *never* use statements within the scope of that assumption later in the proof.

You can use CP repeatedly within the same proof, whether nested or sequentially.
This is a nested CP:

1. $P \supset (Q \vee R)$		
2. $(S \cdot P) \supset \sim Q$	/ $(S \supset P) \supset (S \supset R)$	
	3. $S \supset P$	ACP	Now we want $S \supset R$.
		4. S	ACP	Now we want R.
		5. P	3, 4, MP	
		6. $Q \vee R$	1, 5, MP	
		7. $S \cdot P$	4, 5, Conj	
		8. $\sim Q$	2, 7, MP	
		9. R	6, 8, DS	
	10. $S \supset R$	4-9, CP	
11. $(S \supset P) \supset (S \supset R)$	3-10, CP	

QED

These are sequential uses. This demonstrates how CP is useful for proving biconditionals.

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1. (B ∨ A) ⊃ D
2. A ⊃ ~D
3. ~A ⊃ B          / B ≡ D
          |4. B              ACP
          |5. B ∨ A          4, Add
          |6. D              1, 5, MP
7. B ⊃ D                     4-6 CP
          |8. D              ACP                Note: We can't use the B from the previous assumption.
          |9. ~A             2, 8, DN, MT
          |10. B             3, 9, MP
11. D ⊃ B                    8-10 CP
12. (B ⊃ D) · (D ⊃ B)        7, 11, Conj
13. B ≡ D                    12, Equiv
QED
```

This should always be your first thought when proving biconditionals:
You want: 'P ≡ Q', which is logically equivalent to '(P ⊃ Q) · (Q ⊃ P)'.
Assume P, Derive Q, Discharge.
Assume Q, Derive P, Discharge.
Conjoin the two conditionals.
Use Material Equivalence to yield the biconditional.
This method does not always work, but it's usually a good first thought.

You may use CP along the way to prove statements which are not your main conclusion:

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1. P ⊃ (Q · R)
2. (P ⊃ R) ⊃ (S · T)         / T
          |3. P              ACP
          |4. Q · R          1, 3, MP
          |5. R              4, Com, Simp
6. P ⊃ R                     3-5, CP
7. S · T                     2, 6, MP
8. T                         7, Com, Simp
QED
```

Lastly, CP is also useful to prove arguments without premises.
If a statement is provable with no premises, then it must be a logical truth.
In propositional logic, the logical truths are all tautologies.

Example: Prove that [P ⊃ (Q ⊃ R)] ⊃ [(P ⊃ Q) ⊃ (P ⊃ R)] is a tautology

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          |1. P ⊃ (Q ⊃ R)                       ACP    (to prove (P ⊃ Q) ⊃ (P ⊃ R))
          |         |2. P ⊃ Q             ACP    (to prove (P ⊃ R))
          |         |         |3. P       ACP    (to prove R)
          |         |         |4. Q ⊃ R   1, 3, MP
          |         |         |5. Q       2, 3, MP
          |         |         |6. R       4, 5, MP
          |         |7. P ⊃ R             3-6 CP
          |8. (P ⊃ Q) ⊃ (P ⊃ R)           2-7, CP
9.  [P ⊃ (Q ⊃ R)] ⊃ [(P ⊃ Q) ⊃ (P ⊃ R)]   1-8, CP
QED
```

Note that the last line is further un-indented than the first line, since the first line is indented.

II. **Exercises**. Derive the conclusions of each of the following arguments using the 18 rules, and the method of conditional proof.

1)
1. $A \supset B$
2. $(A \cdot B) \supset D$ / $A \supset D$

2)
1. $H \supset (E \supset F)$
2. $H \supset (G \supset F)$
3. $\sim F$ / $H \supset \sim(E \vee G)$

3)
1. $\sim L \supset M$
2. $\sim(L \cdot M)$ / $\sim M \equiv L$

4)
1. $K \supset (G \vee \sim I)$
2. $I \supset (G \supset J)$ / $K \supset (I \supset J)$

5)
1. $A \supset (B \vee D)$
2. $E \supset (\sim D \supset P)$
3. $\sim D$ / $\sim(B \vee P) \supset \sim(A \vee E)$

6)
Prove that '$(P \supset Q) \supset [(P \cdot R) \supset (Q \cdot R)]$ is a logical truth.

7)
Prove that $(P \cdot Q) \supset [(P \vee R) \cdot (Q \vee R)]$ is a logical truth.

Solutions may vary.