Philosophy 240: Symbolic Logic

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Class #41 - Second-Order Quantification

Second-Order Inferences

- Consider a red apple and a red fire truck.
  (∃x)(Rx • Ax)
  (∃x)(Rx • Fx)
- We might want to infer that they have something in common, that they share a property.

1.	(∃x)(Rx • Ax)	
2.	(∃x)(Rx • Fx)	
3.	Ra • Aa	1, EI
4.	Rb • Ab	3, EI
5.	Ra	3, Simp
6.	Rb	4, Simp
7.	Ra • Rb5,	6, Conj
8.	(∃X)(Xa • Xb)	7, by existential generalization over predicates

Predicate Variables

- In the prior slide, I treated the predicate ‘R’ as subject to quantification, like a singular term.
- A language which allows quantification over predicate places is called a second-order language.
- A system of logic which uses a second-order language is called second-order logic.
  - We’ll call our second-order logic **S**.
- Second-order logic is controversial.
  - Let’s look at it first.
  - Then we can talk about the controversy.

Vocabulary of S

- Capital letters
  - A...U, used as predicates
  - **V, W, X, Y, and Z, used as predicate variables**
- Lower case letters
  - a, b, c, d, e, i, j, k...u are used as constants.
  - f, g, and h are used as functors.
  - v, w, x, y, z are used as singular variables.
- Five connectives: ~, •, ∨, ⊃ ≡
- Quantifiers: ∃, ∀
- Punctuation: (), [], {}

Formation Rules for Wffs of S

1. An n-place predicate or predicate variable followed by n terms (constants, variables, or functor terms) is a wff.

2. For any singular variable β, if α is a wff that does not contain either ‘(∃β)’ or ‘(∀β)’, then ‘(∃β)α’ and ‘(∀β)α’ are wffs.

3. **For any predicate variable β, if α is a wff that does not contain either ‘(∃β)’ or ‘(∀β)’, then ‘(∃β)α’ and ‘(∀β)α’ are wffs.**

4. If α is a wff, so is ∼α.

5. If α and β are wffs, then so are:

- (α · β)
- (α ∨ β)
- (α ⊃ β)
- (α ≡ β)

6. These are the only ways to make wffs.

Uses of Predicate Variables

- No two distinct things have all properties in common.
  - $(\forall x)(\forall y)[x \neq y \supset (\exists X)(Xx \bullet \sim Xy)]$
- Identical objects share all properties (Leibniz's law).
  - $(\forall x)(\forall y)[x=y \supset (\forall Y)(Yx \equiv Yy)]$
- The identity of indiscernibles
  - $(\forall x)(\forall y)[(\forall Z)(Zx \equiv Zy) \supset x=y]$
- The Law of the Excluded Middle
  - $(\forall X)(X \vee \sim X)$
- Analogies: Cat is to meow as dog is to bark.
  - $(\exists X)(Xcm \bullet Xdb)$
- First-order mathematical induction schema can be written as an axiom.
  - $(\forall X)\{\{Na \bullet Xa \bullet (\forall x)[(Nx \bullet Xx) \supset Xf(x)]\} \supset (\forall x)(Nx \supset Xx)\}$
  - a: zero
  - Nx: x is a number

More Translations

1. Everything has some relation to itself.
   - $(\forall x)(\exists V)Vxx$
2. All people have some property in common.
   - $(\forall x)(\forall y)[(Px \bullet Py) \supset (\exists Y)(Yx \bullet Yy)]$
3. No two people have every property in common.
   - $(\forall x)(\forall y)[(Px \bullet Py) \supset (\exists Z)(Zx \bullet {\sim}Zy)$

Characterizing Relations

- We can regiment basic characteristics of relations without second-order logic.
- Here are three characteristics of relations, in first-order logic:
  - Reflexivity: $(\forall x)Rxx$
  - Symmetry: $(\forall x)(\forall y)(Rxy \equiv Ryx)$
  - Transitivity: $(\forall x)(\forall y)(\forall z)[(Rxy \bullet Ryz) \supset Rxz]$
  - Asymmetry: $(\forall x)(\forall y)(Rxy \equiv {\sim}Ryx)$
  - Antisymmetry: $(\forall x)(\forall y)[(Rxy \bullet Ryx) \supset x=y]$
    - Asymmetry entails antisymmetry
    - $<$ is asymmetric
    - $\leq$ is antisymmetric

Characterizing Relations in Second- Order Logic

- Second-order logic allows us to do more.
- Some relations are transitive.
  - (∃X)(∀x)(∀y)(∀z)[(Xxy • Xyz) ⊃ Xxz]
- Some relations are symmetric, while some are asymmetric.
  - (∃X)(∀x)(∀y)(Xxy ≡ Xyx) • (∃X)(∀x)(∀y)(Xxy ≡ ~Xyx)

Replacing the Identity Predicate

$$x=y \text{ iff } (\forall X)(Xx \equiv Xy)$$

- Symmetry and reflexivity follow from the symmetry and reflexivity of the biconditional.
- IDi follows from BMP.

Identity for Properties

- We would like to say something about property identity.
  - For example: There are at least two distinct properties.
  - $(\exists X)(\exists Y)X \neq Y$
- But identity is a relation between singular terms, not predicates.
- And there are no objects attached to the predicates above.
- We can add a quantifier to take care of the singular terms:
  - $(\exists X)(\exists Y)(\exists x){\sim}(Xx \equiv Yx)$
- This only indicates that there are distinct monadic properties.
- What about dyadic properties?
  - $(\exists X)(\exists Y)(\exists x)(\exists y){\sim}(Xxy \equiv Yxy)$
- In order to generalize such claims, higher-order logics are required.

Higher-Order Logics

- All logics after first-order logic are called higher-order logic
- To create third-order logic, we introduce attributes of attributes.
- All useful properties are desirable.
  - $(\forall X)(\boldsymbol{U}X \supset \boldsymbol{D}X)$
  - Not a wff in **S**; it lacks singular terms
- A man who possesses all virtues is a virtuous man, but there are virtuous men who do not possess all virtues:
  - $(\forall x)\{[Mx \bullet (\forall X)(\boldsymbol{V}X \supset Xx)] \supset \boldsymbol{V}x\} \bullet (\exists x)[Mx \bullet Vx \bullet (\exists X)(\boldsymbol{V}X \bullet {\sim}Xx)]$
  - More missing singular terms.
  - Also, the third-order variables are applied both to predicates and terms, which is a category error.
- Cleaning it up would make it messier.
  - Yes, I said that.