Philosophy 240: Symbolic Logic

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Class #29
Semantics for Predicate Logic

Theories

- A *theory* is a set of sentences.
- A formal theory is a set of sentences of a formal language.
- We identify a theory by its theorems, the set of sentences provable within that theory.
- Many interesting formal theories are infinite.
  - Rules of inference generate an infinite number of theorems.

Constructing Formal Theories

1. Specify a language
   - vocabulary
   - formation rules for wffs

Syntax for **PL** and **M**

Vocabulary for **PL**

Capital letters A...Z
Five connectives: ~, •, ∨, ⊃ ≡
Punctuation: ( ), [ ], { }

Formation rules for Wffs of **PL**

1. A single capital English letter is a wff.
2. If α is a wff, so is ~α.
3. If α and β are wffs, then so are:
   (α · β)
   (α ∨ β)
   (α ⊃ β)
   (α ≡ β)
4. These are the only ways to make wffs.

Vocabulary for **M**

Capital letters A...Z used as one-place predicates
Lower case letters used as singular terms
  a, b, c,...u are constants.
  v, w, x, y, z are variables.
Five connectives: ~, •, ∨, ⊃ ≡
Quantifier symbols: ∃, ∀
Punctuation: ( ), [ ], { }

Formation Rules for Wffs of **M**

1. A predicate (capital letter) followed by a constant or variable (lower-case letter) is a wff.
2. For any variable β, if α is a wff that does not contain either '(∃β)' or '(∀β)', then '(∃β)α' and '(∀β)α' are wffs.
3. If α is a wff, so is ~α.
4. If α and β are wffs, then so are:
   (α • β)
   (α ∨ β)
   (α ⊃ β)
   (α ≡ β)
5. These are the only ways to make wffs.

Constructing Formal Theories

1. Specify a language
   - vocabulary
   - formation rules for wffs
2. Add formation rules for wffs.

To construct a formal theory, we select some of the wffs as our theorems.

- Different theories can be written in the same language.

3a. Specify theorems

- We can list all of our theorems: finite theories
- We can specify axioms and rules of inference.
  - Proof Theory

3b. Provide a semantics for the theory.

- Specify truth conditions and truth values for wffs
- Model Theory (Truth)

Goodness for Theories

- In some theories, the provable theorems are exactly the same as the true wffs.
  - Proof theory and semantics align!
  - *Soundness*: all the provable theorems are true
  - *Completeness*: all the truths are provable
- In more sophisticated theories, proof separates from truth
  - Gödel's first incompleteness theorem
  - For most interesting theories beyond PL and M, there are true sentences of the theory that are not provable within the system
  - Model theory and proof theory come apart.
  - This is a mind-blowingly awesome result.

Semantics and Proof Theory for **PL**

- In **PL**, our semantics used truth tables.
  - Interpretations of **PL**
    - **Step 1. Assign 1 or 0 to each atomic sentence.**
      - Only finitely many (2^{26} = ~6.7 million) possible interpretations in our language.
      - We could use a language with infinitely many simple terms: P, P’, P’’, P’’’, P’’’’...
    - **Step 2. Assign truth values to complex propositions by combining, according to the truth table definitions, the truth values of the atomic sentences.**
  - In **M**, and the other languages of predicate logic, the semantics are more complicated.
    - interpretation, satisfaction, logical truth, validity
- In proof theory, we construct a system of inference using the formal language we have specified.
  - In **PL**, our proof system was our twenty-four rules of natural deduction, plus conditional and indirect proof.
  - Other proof systems use axioms.

Semantics for M

- Separating the syntax of our language from its semantics allows us to treat our formal languages as completely uninterpreted.
  - Intuitively, we know what the logical operators mean.
  - But until we specify a formal interpretation, we are free to interpret them as we wish.
- Our constants and predicates and quantifiers are, as far as the syntax of our language specifies, uninterpreted.
- To look at the logical properties of the language, we construct formal semantics.
- The first step in formal semantics is to show how to provide an interpretation of the language.
- Then, we can determine the logical truths.
  - The wffs that come out as true under every interpretation.

Interpretations of M

- To define an interpretation in M, or in any of its extensions, we have to specify how to handle constants, predicates, and quantifiers.
  - We use some set theory in our meta-language.
- Step 1. Specify a set to serve as a domain of interpretation (or quantification).
  - We can consider small finite domains

    Domain_1 = {1, 2, 3}

    Domain_2 = {Barack Obama, Hillary Clinton, Joe Biden}.
  - We can consider larger domains, like a universe of everything.
- Step 2. Assign a member of the domain to each constant.

    a: 1; b: 2; c: 3

    a: Obama; b: Clinton
- Step 3. Assign some set of objects in the domain to each predicate.
  - 'Ex' may stand for 'x has been elected president'
  - In Domain_1, the interpretation of 'Ex' will be empty.
  - In Domain_2, it will be {Barack Obama}.
- Step 4. Use the customary truth tables for the interpretation of the connectives.

Satisfaction and Truth-for-an-Interpretation

- Objects in the domain may satisfy predicates.
  - Ordered n-tuples may satisfy relations.
- A wff will be satisfiable if there are objects in the domain of quantification which satisfy the predicates indicated in the wff.
  - A universally quantified sentence is satisfied if it is satisfied by all objects in the domain.
  - An existentially quantified sentence is satisfied if it is satisfied by some object in the domain.
- A wff will be true-for-an-interpretation if all objects in the domain of quantification satisfy the predicates indicated in the wff.
- We call an interpretation on which all of a set of statements come out true a *model*.

An Interpretation of a Theory

- Theory
  1. Pa • Pb
  2. Wa • ~Wb
  3. (∃x)Px
  4. (∀x)Px
  5. (∀x)(Wx ⊃ Px)
  6. (∀x)(Px ⊃ Wx)
- Step 1: Specify a set to serve as a domain of interpretation, or domain of quantification.
  - Domain: {Bob Simon, Rick Werner, Katheryn Doran, Todd Franklin, Marianne Janack, Russell Marcus, Theresa Lopez, Alex Plakias, Doug Edwards}
- Step 2: Assign a member of the domain to each constant.
  - a: Katheryn Doran
  - b: Bob Simon

    Notice: no other constants in our theory

    Some objects remain without names
- Step 3: Assign some set of objects in the domain to each predicate.
  - Px: {Bob Simon, Rick Werner, Katheryn Doran, Todd Franklin, Marianne Janack, Russell Marcus, Theresa Lopez, Alex Plakias, Doug Edwards}
  - Wx: {Katheryn Doran, Marianne Janack, Theresa Lopez, Alex Plakias}

Models in M

1. Pa • Pb
2. Wa • ~Wb
3. (∃x)Px
4. (∀x)Px
5. (∀x)(Wx ⊃ Px)
6. (∀x)(Px ⊃ Wx)

Domain: {Bob Simon, Rick Werner, Katheryn Doran, Todd Franklin, Marianne Janack, Russell Marcus, Theresa Lopez, Alex Plakias, Doug Edwards}
a: Katheryn Doran
b: Bob Simon
Px: {Bob Simon, Rick Werner, Katheryn Doran, Todd Franklin, Marianne Janack, Russell Marcus, Theresa Lopez, Alex Plakias, Doug Edwards}
Wx: {Katheryn Doran, Marianne Janack, Theresa Lopez, Alex Plakias}

- We call an interpretation on which all of a set of given statements come out true a *model*.
- Given our interpretations of the predicates, not every sentence in our set is satisfied.
  - 1-5 are satisfied.
  - 6 is not.
- If we were to delete sentence 6 from our list, our interpretation would be a model.

Constructing a Model

- Theory
  1. (∀x)(Px ⊃ Qx)
  2. (∃x)(Px • Rx)
  3. (∃x)(Qx • ~Px)
  4. (∃x)(Qx • ~Rx)
  5. (Pa • Pb) • Qc
- Step 1. Specify a set to serve as a domain of interpretation, or domain of quantification.
  Domain = {Persons}

- **Step 2**. Assign a member of the domain to each constant.
  a = Barack Obama
  b = Condoleezza Rice
  c = Neytiri (from *Avatar*)
- **Step 3**. Assign some set of objects in the domain to each predicate.
  Px = {Human Beings}
  Qx = {Persons}
  Rx = {Males}
- **Step 4**. Use the customary truth tables for the interpretation of the connectives.

Logical Truth in M

- A wff of M will be logically true if it is true for every interpretation.
- For **PL**, the notion of logical truth was simple.
  - Just look at the truth tables.
- For **M**, and even more so for **F** (full first-order logic), the notion of logical truth is naturally complicated by the fact that we are analyzing parts of propositions.
- Here are two logical truths of **M**:
  - $(\forall x)(Px \vee {\sim}Px)$
  - $Pa \vee [(\forall x)Px \supset Qa]$
- As in **PL**, we can show that a wff is a theorem (logical truth) proof-theoretically and model-theoretically.

Proof-Theoretic Argument

$(\forall x)(Px \vee {\sim}Px)$

1. $\sim(\forall x)(Px \vee {\sim}Px)$	AIP
2. $(\exists x){\sim}(Px \vee {\sim}Px)$	1, QE
3. $\sim(Pa \vee {\sim}Pa)$	2, EI
4. $\sim Pa \bullet {\sim}{\sim}Pa$	3, DM
5. $(\forall x)(Px \vee {\sim}Px)$	1-4, IP, DN

Model-Theoretic Argument

$$Pa \vee [(\forall x)Px \supset Qa]$$

- Consider an interpretation on which ‘$Pa \vee [(\forall x)Px \supset Qa]$’ is false.
- The object assigned to ‘a’ will not be in the set assigned to ‘Px’, and there is some counterexample to ‘$(\forall x)Px \supset Qa$’.
- But, any counter-example to a conditional statement has to have a true antecedent.
- So, every object in the domain will have to be in the set assigned to ‘Px’.
  - Tilt
- So, no interpretation will make that sentence false.
- So, ‘$Pa \vee [(\forall x)Px \supset Qa]$’ is logically true.

Another Logical Truth

$$(\exists x)Px \vee (\forall x)(Qx \supset \sim Px)$$

- Try it both ways!

Validity

- A valid argument will have to be valid under any interpretation, using any domain.
- Our proof system has given us ways to show that an argument is valid.
- But when we introduced our system of inference for **PL**, we already had a way of distinguishing the valid from the invalid arguments, using truth tables.
- In **M**, we need a corresponding method for showing that an argument is invalid.
- An invalid argument will have counter-examples, interpretations on which the premises come out true and the conclusion comes out false.

Coming Up

- Monday
  - Invalid Arguments in Predicate Logic
  - Constructing Counterexamples
- But first: Friday
  - Test #4