Philosophy 240
Symbolic Logic

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Fall 2015

Class #24: Infinity

Business

- HW due today?

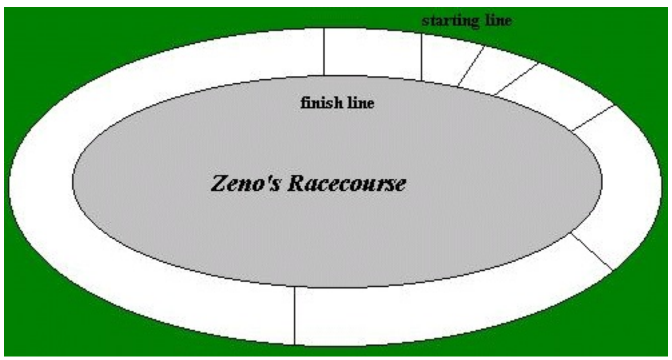
Nineteenth Century Developments in Mathematics

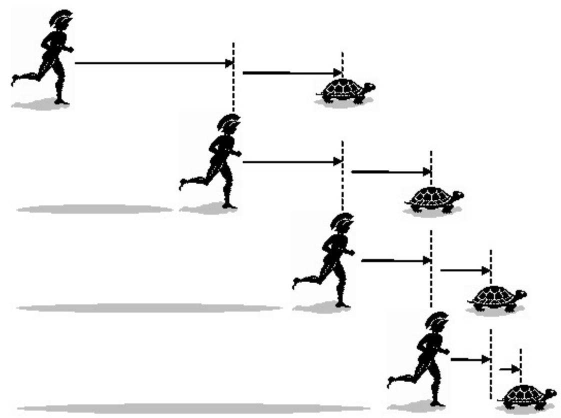
worries about logical entailments

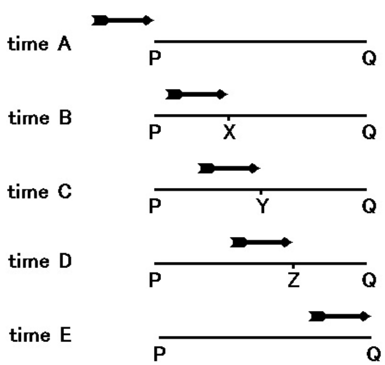
- Non-Euclidean geometries
- The calculus of infinitesimals
- Cantor's proof that there are different sizes of infinity
- Frege developed logic to show that the proofs in such cases were legitimate.
- So the development of formal logic and questions about infinite are historically linked.
- But questions about infinity go back a long way before the nineteenth century.

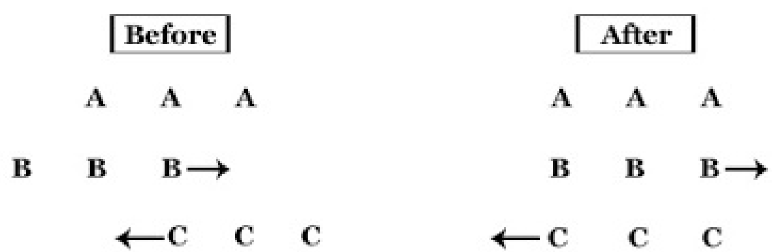
Zeno's Paradoxes

- Achilles and the Racetrack
- Achilles and the Tortoise
- The Arrow
- Moving Blocks





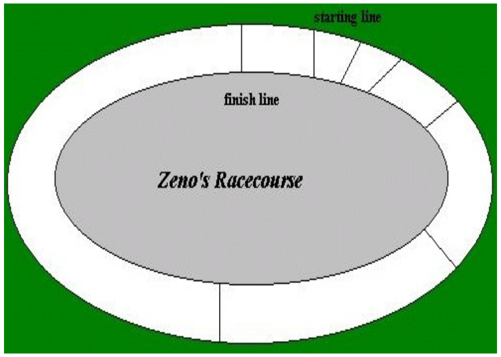


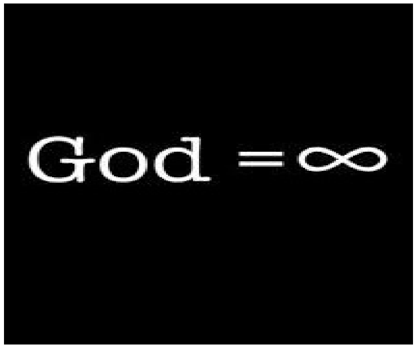


Aristotle

Potential Infinite and Actual Infinity

- We could divide the racetrack into infinitely many parts.
- But we don't.
- There is no actual infinity in the world.
- So, Zeno's paradoxes are no problem.
- Medievals and Moderns identified infinity with God.

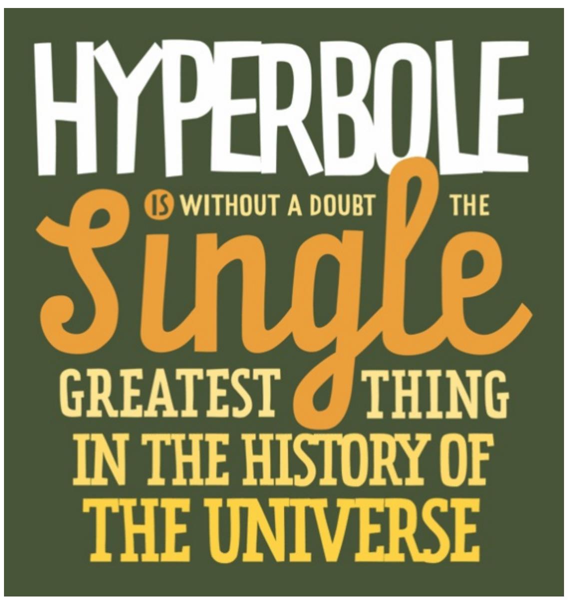




Cantor

Infinity is more complicated than we thought

- Mid Nineteenth Century
- Diagonal Argument
- The most important intellectual achievement of the nineteenth century!



The Infinite Hotel

- One guest
- Any finite number of guests
- An infinite busload of guests
- Infinitely many infinite busloads of guests
- Are there guests that can't be accommodated?

Two Concepts of Size

Cardinal Numbers

- $Size_H$: One-One Correspondence
- $Size_W$: Whole is Greater than its Parts
- For finite sets, $size_H$ and $size_W$ are the same.
- For infinite sets, they diverge.
  - The natural numbers and the even natural numbers have the same $size_H$ but different $size_W$s.
  - An infinite set can be put into one-one correspondence with a proper subset of itself.

Lists of Numbers

- Natural Numbers
- Integers (positive and negative)
- Rationals

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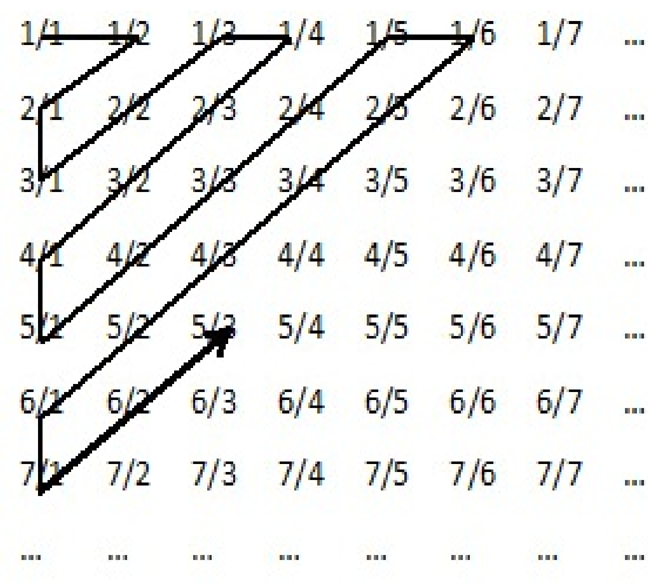

This is not the diagonal argument!

The Diagonal Argument

Are there sets of numbers we could not list?

- Cantor: Yes!
  - The real numbers
- Consider the real numbers in their decimal expansions.
  - The same argument can work in different bases, or with sets, or any way the numbers can be presented.
- Let's suppose (for indirect proof) that there is such a list:
  - $a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7$...
  - $b_1\ b_2\ b_3\ b_4\ b_5\ b_6\ b_7$...
  - $c_1\ c_2\ c_3\ c_4\ c_5\ c_6\ c_7$...
  - $d_1\ d_2\ d_3\ d_4\ d_5\ d_6\ d_7$...
  - ...
- Consider the number N
  - $a_1\ b_2\ c_3\ d_4\ e_5\ f_6\ g_7$...
  - Could be on the list
- Construct a new number N*
  - $a_1^*\ b_2^*\ c_3^*\ d_4^*\ e_5^*\ f_6^*\ g_7^*$...
  - Use the diagonal
  - Not on the list

Cardinal Arithmetic 1

Properties of all cardinals

1. $a+b=b+a$
2. $ab=ba$
3. $a + (b + c) = (a + b) + c$
4. $a \bullet (b \bullet c) = (a \bullet b) \bullet c$
5. $a \bullet (b + c) = ab + ac$
6. $a^{(b+c)} = a^b \bullet a^c$
7. $(ab)^c = a^c \bullet b^c$
8. $(a^b)^c = a^{bc}$

Cardinal Arithmetic 2

Where finites and transfinites diverge

- For transfinites:
  - a + 1 = a
  - 2a = a
  - a • a = a
- Recall the infinite hotel
  - Adding one guest
    - (or any finite number or guests)
  - Adding an infinite bus
  - Adding an infinite number of infinite buses

Cardinal Arithmetic 3

Power sets and Exponentiation

- $2^a > a$
  - for both finite and infinite numbers
- Cantor's Theorem

Infinite Counting

- Call the size of the natural numbers $\aleph_0$
- Then:
  - $\aleph_0, \aleph_1, \aleph_2, \aleph_3, \aleph_4 \ldots$
- What is the size of $2^{\aleph_0}$?
  - 2^3 does not give you the next natural number after 2^2.
  - Continuum Hypothesis
  - What is the structure of the transfinite numbers?
- That’s (at least in part) what motivates Frege to develop modern logic.