Philosophy 240
Symbolic Logic

Russell Marcus
Hamilton College
Fall 2015

Class #20: On Axiomatic Systems

Business

- Almost halfway!
- Test #3 on Wednesday, on Chapter 2
  - Biconditional rules
  - CP
  - IP
  - Logical truths
- Predicate logic starts right after break
- Today:
  - A bit on axiomatic systems
  - Practice problems
  - Solutions are already on line

Hunter's System PS

With axiom schemata

- Three axiom schemata:

  PS1: $\alpha \supset (\beta \supset \alpha)$

  PS2: $(\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$

  PS3: $(\sim\alpha \supset \sim\beta) \supset (\beta \supset \alpha)$

  Any formula of the language PS of any of the forms PS1, PS2, or PS3 is an axiom of PS.

  – Infinitely many axioms of PS

- One rule of inference, modus ponens.

  If α and β are formulas, then β is a consequence in system PS of α and $\alpha \supset \beta$.

  $\alpha, (\alpha \supset \beta) \vdash_{PS} \beta$

Hunter's System PS

With a substitution rule

- Three axioms

  PS1*: $P \supset (Q \supset P)$

  PS2*: $(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$

  PS3*: $(\sim P \supset \sim Q) \supset (Q \supset P)$

- Two rules of inference

  MP: $\alpha, (\alpha \supset \beta) \vdash_{PS} \beta$

  Substitution: For any wff α, and any wff β appearing as a sub-formula in wff δ: If $\vdash_{PS} \delta$, then $\vdash_{PS} \delta_{\alpha}^{\beta}$

  – where 'δ_{α}^{β}' is the result of substituting 'α' for 'β' throughout.

Hunter’s PS and Our PL

- Hunter’s axiomatic system is provably equivalent to our more-familiar natural deduction system PL.
  - The deduction theorem allows us to move between the two kinds of systems.
- Frege’s original *Begriffsschrift* used a Hilbert-style axiomatic system, with a different axiomatization.
  - See the appendices in Richard Mendelsohn’s *The Philosophy of Gottlob Frege*, for a translation of the *Begriffsschrift* into modern notation.
- Natural deduction systems like PS in *What Follows* are due largely to Gerhard Gentzen’s work in the 1930s and 1940s.
  - There is a paper on the history of natural deduction by Pelletier which might be worth a look for a term paper.
- Completeness:
  - Frege’s axiomatization, Hunter’s PS, and standard systems of natural deduction are all complete.
    - All logical truths are provable within the system.
- One’s choice of system is thus arbitrary among the various complete systems.

A Derivation in PS

PS1: $\alpha \supset (\beta \supset \alpha)$
PS2: $(\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$
PS3: $(\sim\alpha \supset \sim\beta) \supset (\beta \supset \alpha)$

- A1: $\vdash_{PS}$ $P \supset P$
  1. $P \supset ((P \supset P) \supset P)$ PS1
  2. $(P \supset ((P \supset P) \supset P)) \supset ((P \supset (P \supset P)) \supset (P \supset P))$ PS2
  3. $(P \supset (P \supset P)) \supset (P \supset P)$ MP, 1, 2
  4. $P \supset (P \supset P)$ PS1
  5. $P \supset P$ 3, 4, MP
- QED

A Further Derivation in PS

PS1: $\alpha \supset (\beta \supset \alpha)$
PS2: $(\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$
PS3: $(\sim\alpha \supset \sim\beta) \supset (\beta \supset \alpha)$
A1: $P \supset P$

- A2: $\vdash_{PS} \sim P \supset (P \supset P)$

1.	$P \supset P$	A1
2.	$(P \supset P) \supset (\sim P \supset (P \supset P))$	PS 1
3.	$\sim P \supset (P \supset P)$	MP, 1, 2

- QED

A Longer Proof in PS

- $\vdash_{PS}$ **(P ⊃ (Q ⊃ R)) ⊃ (Q ⊃ (P ⊃ R))**
  1. Q ⊃ (P ⊃ Q)
  2. (P ⊃ (Q ⊃ R)) ⊃ ((P ⊃ Q) ⊃ (P ⊃ R))
  3. [(P ⊃ Q) ⊃ (P ⊃ R)] ⊃ [Q ⊃ ((P ⊃ Q) ⊃ (P ⊃ R))]
  4. [((P ⊃ Q) ⊃ (P ⊃ Q')) ⊃ (Q ⊃ ((P ⊃ Q) ⊃ (P ⊃ R)))] ⊃ {(P ⊃ (Q ⊃ R)) ⊃ [((P ⊃ Q) ⊃ (P ⊃ R)) ⊃ (Q ⊃ ((P ⊃ Q) ⊃ (P ⊃ R)))]}
  5. (P ⊃ (Q ⊃ R)) ⊃ [((P ⊃ Q) ⊃ (P ⊃ R)) ⊃ (Q ⊃ ((P ⊃ Q) ⊃ (P ⊃ R)))]
  6. {(P ⊃ (Q ⊃ R)) ⊃ [((P ⊃ Q) ⊃ (P ⊃ R)) ⊃ (Q ⊃ ((P ⊃ Q) ⊃ (P ⊃ R)))]} ⊃ {[(P ⊃ (Q ⊃ R)) ⊃ ((P ⊃ Q) ⊃ (P ⊃ R))] ⊃ [(P ⊃ (Q ⊃ R)) ⊃ (Q ⊃ ((P ⊃ Q) ⊃ (P ⊃ R)))]}
  7. [(P ⊃ (Q ⊃ R)) ⊃ ((P ⊃ Q) ⊃ (P ⊃ R))] ⊃ [(P ⊃ (Q ⊃ R)) ⊃ (Q ⊃ ((P ⊃ Q) ⊃ (P ⊃ R)))]
  8. (P ⊃ (Q ⊃ R)) ⊃ [Q ⊃ ((P ⊃ Q) ⊃ (P ⊃ R))]
  9. [Q ⊃ ((P ⊃ Q) ⊃ (P ⊃ R))] ⊃ [(Q ⊃ (P ⊃ Q)) ⊃ (Q ⊃ (P ⊃ R))]
  10. [(Q ⊃ ((P ⊃ Q) ⊃ (P ⊃ R))) ⊃ ((Q ⊃ (P ⊃ Q)) ⊃ (Q ⊃ (P ⊃ R)))] ⊃ {(P ⊃ (Q ⊃ R)) ⊃ [(Q ⊃ ((P ⊃ Q) ⊃ (P ⊃ R))) ⊃ ((Q ⊃ (P ⊃ Q)) ⊃ (Q ⊃ (P ⊃ R)))]}
  11. (P ⊃ (Q ⊃ R)) ⊃ [(Q ⊃ ((P ⊃ Q) ⊃ (P ⊃ R))) ⊃ ((Q ⊃ (P ⊃ Q)) ⊃ (Q ⊃ (P ⊃ R)))]
  12. {(P ⊃ (Q ⊃ R)) ⊃ [(Q ⊃ ((P ⊃ Q) ⊃ (P ⊃ R))) ⊃ ((Q ⊃ (P ⊃ Q)) ⊃ (Q ⊃ (P ⊃ R)))]} ⊃ {[(P ⊃ (Q ⊃ R)) ⊃ (Q ⊃ ((P ⊃ Q) ⊃ (P ⊃ R)))] ⊃ [(P ⊃ (Q ⊃ R)) ⊃ ((Q ⊃ (P ⊃ Q)) ⊃ (Q ⊃ (P ⊃ R)))]}
  13. [(P ⊃ (Q ⊃ R)) ⊃ (Q ⊃ ((P ⊃ Q) ⊃ (P ⊃ R)))] ⊃ [(P ⊃ (Q ⊃ R)) ⊃ ((Q ⊃ (P ⊃ Q)) ⊃ (Q ⊃ (P ⊃ R)))]
  14. (P ⊃ (Q ⊃ R)) ⊃ [(Q ⊃ (P ⊃ Q)) ⊃ (Q ⊃ (P ⊃ R))]
  15. {(P ⊃ (Q ⊃ R)) ⊃ [(Q ⊃ (P ⊃ Q)) ⊃ (Q ⊃ (P ⊃ R))]} ⊃ {[(P ⊃ (Q ⊃ R)) ⊃ (Q ⊃ (P ⊃ Q))] ⊃ [(P ⊃ (Q ⊃ R)) ⊃ (Q ⊃ (P ⊃ R))]}
  16. [(P ⊃ (Q ⊃ R)) ⊃ (Q ⊃ (P ⊃ Q))] ⊃ [(P ⊃ (Q ⊃ R)) ⊃ (Q ⊃ (P ⊃ R))]
  17. (Q ⊃ (P ⊃ Q)) ⊃ [(P ⊃ (Q ⊃ R)) ⊃ (Q ⊃ (P ⊃ Q))]
  18. (P ⊃ (Q ⊃ R)) ⊃ (Q ⊃ (P ⊃ Q))
  19. (P ⊃ (Q ⊃ R)) ⊃ (Q ⊃ (P ⊃ R))
- QED