

Philosophy 240
Symbolic Logic

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Class #13: Modal Logic

Business

- No more missed classes for me
- You have 13 rules
 - Double negation
 - Derivations are getting harder
- Next week
 - Monday: 5 more rules
 - Wednesday: 7 more rules
 - Friday: Test #2
- Today
 - A logical enrichment topic (technical)
 - Profound philosophical impact, especially since the 1970s.

Sentential Operators

- In **PL** (and **M**), we have four binary sentential operators.
- We also have a unary sentential operator: negation.
- There are other ways to modify sentences.
- Negation
 - ▶ It is not the case that the sun is shining.
- Modal Operators
 - ▶ It is possible that the sun is out.
 - It is possibly the case that the sun is out.
 - The sun might be out.
 - ▶ It is necessary that the sun is out.
 - It is necessarily the case that the sun is out.
 - The sun is definitely (surely, certainly) out.

Propositional Modal Logic (PML)

Formation Rules

1. A single capital English letter is a wff.
2. If α is a wff, so is $\sim\alpha$.
3. If α and β are wffs, then so are $(\alpha \cdot \beta)$, $(\alpha \vee \beta)$, $(\alpha \supset \beta)$, and $(\alpha \equiv \beta)$.
4. If α is a wff, then so are $\diamond\alpha$ and $\Box\alpha$.
5. These are the only ways to make wffs.

The Relation between \diamond and \square

- $\square\alpha \Leftrightarrow \sim\diamond\sim\alpha$
- $\diamond\alpha \Leftrightarrow \sim\square\sim\alpha$
- We'll see a similar structure with quantifiers when we get to predicate logic.

Interpretations of \diamond and \square

$$\diamond\alpha \Leftrightarrow \sim\square\sim\alpha$$

$$\square\alpha \Leftrightarrow \sim\diamond\sim\alpha$$

- Alethic
 - 'It is possible that' and 'It is necessary that'
- Deontic
 - 'It is morally permissible that' and 'it is morally required that'
- Temporal
 - 'At some time in the future' and 'at all times in the future'
- Dynamic
 - 'after some change' and 'after any change'
 - 'for some state' and 'for every state'
- Epistemic
 - Epistemic: \square is 'it is known that'
 - Doxastic: \square is 'it is believed that'
- Exploring any of these interpretations, or others, would be a wonderful paper topic.

Actual World Semantics

Just the same as the truth tables

$\mathcal{V}(\sim\alpha) = 1$ if $\mathcal{V}(\alpha) = 0$; otherwise $\mathcal{V}(\sim\alpha) = 0$

$\mathcal{V}(\alpha \bullet \beta) = 1$ if $\mathcal{V}(\alpha) = 1$ and $\mathcal{V}(\beta) = 1$; otherwise $\mathcal{V}(\alpha \bullet \beta) = 0$

$\mathcal{V}(\alpha \vee \beta) = 1$ if $\mathcal{V}(\alpha) = 1$ or $\mathcal{V}(\beta) = 1$; otherwise $\mathcal{V}(\alpha \bullet \beta) = 0$

$\mathcal{V}(\alpha \supset \beta) = 1$ if $\mathcal{V}(\alpha) = 0$ or $\mathcal{V}(\beta) = 1$; otherwise $\mathcal{V}(\alpha \supset \beta) = 0$

$\mathcal{V}(\alpha \equiv \beta) = 1$ if $\mathcal{V}(\alpha) = \mathcal{V}(\beta)$; otherwise $\mathcal{V}(\alpha \equiv \beta) = 0$

Possible World Semantics (Leibnizian)

- $\mathcal{V}(\Box\alpha) = 1$ if $\mathcal{V}(\alpha, w_n) = 1$ for all w_n in \mathcal{U}
- $\mathcal{V}(\Box\alpha) = 0$ if $\mathcal{V}(\alpha, w_n) = 0$ for any w_n in \mathcal{U}
- $\mathcal{V}(\Diamond\alpha) = 1$ if $\mathcal{V}(\alpha, w_n) = 1$ for any w_n in \mathcal{U}
- $\mathcal{V}(\Diamond\alpha) = 0$ if $\mathcal{V}(\alpha, w_n) = 0$ for all w_n in \mathcal{U}

$\mathcal{U} = \{w_1, w_2, w_3\}$

At w_1 , P, Q, R and S are all true.

At w_2 , P and Q are true, but R and S are false.

At w_3 , P is true, and Q, R, and S are false.

1. $\Box(P \supset Q)$
2. $\Diamond(P \supset Q)$
3. $\Box P \supset \Box Q$
4. $\Diamond P \supset \Diamond Q$
5. $\Diamond[(Q \vee \sim R) \supset \sim P]$
6. $\Diamond P \supset [Q \supset \Box(R \cdot S)]$

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$\mathcal{U} = \{w_1, w_2, w_3\}$

At w_1 , P, Q, R and S are all true.

At w_2 , P and Q are true, but R and S are false.

At w_3 , P is true, and Q, R, and S are false.

False	1. $\Box(P \supset Q)$
True	2. $\Diamond(P \supset Q)$
False	3. $\Box P \supset \Box Q$
True	4. $\Diamond P \supset \Diamond Q$
False	5. $\Diamond[(Q \vee \sim R) \supset \sim P]$
True at w_3 ,	6. $\Diamond P \supset [Q \supset \Box(R \cdot S)]$
False otherwise	

Different Kinds of Possibility

- Leibnizian semantics is the modal logic of logical possibility.
 - (Called **S5** by C.I. Lewis)
- Logical possibility is very weak.
 - It is logically possible for a bachelor to be married,
 - ...for objects to travel faster than the speed of light,
 - ...for a square to have five sides.
- Semantic possibility
- Physically possibility
- Mathematical possibility

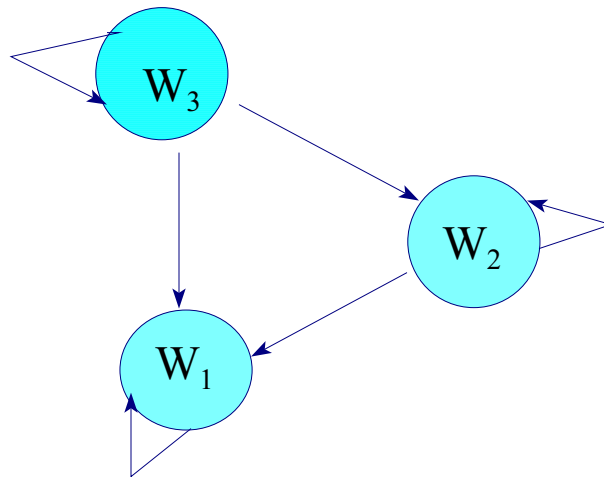
Worlds with Different Possibilities

- Facts vary among worlds.
- Possibility might also vary among worlds.
- World One: ours.
- World Two: like ours, except that there is some force which moves the planets in perfectly circular orbits.
- The law that says that all planets move in elliptical orbits holds in both worlds, since a circle is just a type of ellipse.
- w_2 obeys all the laws of w_1 .
- w_1 does not obey all the laws of w_2 .

Possible World Semantics (Kripkean)

Adds Accessibility Relations

$$\mathcal{R} = \{ \langle w_1, w_1 \rangle, \langle w_2, w_1 \rangle, \langle w_2, w_2 \rangle, \langle w_3, w_1 \rangle, \langle w_3, w_2 \rangle, \langle w_3, w_3 \rangle \}$$



Possible World Semantics (Kripkean)

Valuation Rules

- $\mathcal{V}(\Box\alpha, w_n) = 1$ if $\mathcal{V}(\alpha, w_m) = 1$ for all w_m in \mathcal{U} such that $\langle w_n, w_m \rangle$ is in \mathcal{R}
- $\mathcal{V}(\Box\alpha, w_n) = 0$ if $\mathcal{V}(\alpha, w_m) = 0$ for any w_m in \mathcal{U} such that $\langle w_n, w_m \rangle$ is in \mathcal{R}
- $\mathcal{V}(\Diamond\alpha, w_n) = 1$ if $\mathcal{V}(\alpha, w_m) = 1$ for any w_m in \mathcal{U} such that $\langle w_n, w_m \rangle$ is in \mathcal{R}
- $\mathcal{V}(\Diamond\alpha, w_n) = 0$ if $\mathcal{V}(\alpha, w_m) = 0$ for all w_m in \mathcal{U} such that $\langle w_n, w_m \rangle$ is in \mathcal{R}

1. $\Box(P \supset Q)_1$
2. $\Box(P \supset Q)_3$
3. $\Diamond\sim(Q \vee R)_1$
4. $\Diamond\sim(Q \vee R)_2$
5. $\Diamond\sim(Q \vee R)_3$
6. $\Box P_1 \supset \Box Q_1$
7. $\Box P_3 \supset \Box Q_3$

Systems of Inference

S5 Axioms and Rules

- PL Any tautology of **PL**
- Duality $\diamond\alpha \equiv \sim\Box\sim\alpha$
- K $\Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)$
- T $\Box\alpha \supset \alpha$
- 5 $\diamond\alpha \supset \Box\diamond\alpha$
- Necessitation $\alpha \vdash \Box\alpha$
- MP $\alpha \supset \beta, \alpha \vdash \beta$

Other Modal Systems

- In **S5**, every world is accessible from every other world.
- Weaker modal logics have weaker accessibility relations and characteristic axioms.
 - ▶ The axioms and the accessibility relations go together.
- **K** is weak
 - ▶ $\Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)$
- **D** is stronger
 - ▶ $\Box\alpha \supset \Diamond\alpha$
 - ▶ Can't derive T ($\Box\alpha \supset \alpha$) in **D**
 - ▶ So **D** is not apt for the alethic interpretation of the operators.
 - ▶ But the deontic interpretation of the characteristic axiom of **D** looks better.

Hintikka and the KK Thesis

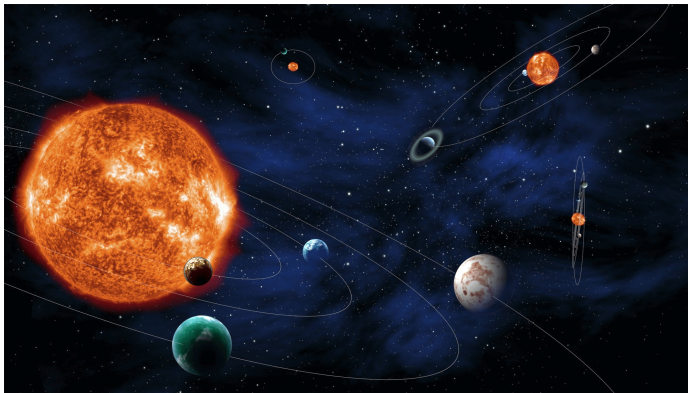
- “Suppose we say that evidence for a proposition, P , is conclusive iff it is so strong that, once one discovers it, further inquiry cannot give one reason to stop believing P . The concept of knowledge used by many philosophers seems to be a strong one on which one knows P only if one’s evidence for P is conclusive in this sense. It is plausible that the KK principle holds for this strong concept of knowledge. For it is plausible that one’s evidence for P is conclusive in the above sense only if it rules out the possibility that one does not know P , and thus only if it allows one to know that one knows P .
- “To see this, suppose one has evidence, E , for a proposition P , and that E does not rule out the possibility that one does not know P . If E does not rule out this possibility, then, after one has discovered E , further inquiry can, in principle, reveal to one that one does not know P . But if further inquiry were to reveal this, then it would surely give one reason to stop believing P (since one should not believe things that one does not know). So it is plausible that, if E does not rule out the possibility that one does not know P , then it is not conclusive in the sense just defined, and hence plausible that, if knowledge requires evidence that is conclusive in this sense, the KK principle holds” (Hintikka 1970: 145-6).

Concerns about Possible Worlds

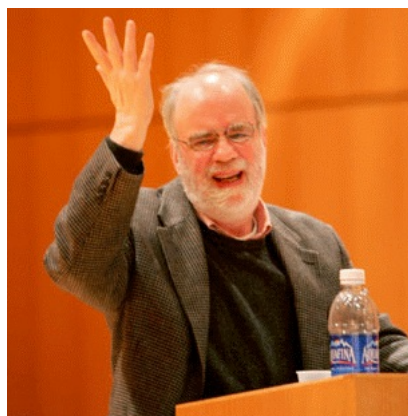
- Metaphysical
 - What is a possible world?
 - Do possible worlds exist?
- Epistemic
 - How do we know about possible worlds?
 - Do we stipulate them?
 - Do we discover them, or facts about them?

A Quinean Concern

- Consider:
 - ▶ A. Nine is greater than seven.
 - ▶ B: The number of planets is greater than seven.
- A and B have the same truth value.
 - ▶ One can be inferred from the other by a simple substitution, given that:
 - ▶ C: The number of planets = nine.
- Now Consider
 - ▶ D: Necessarily, nine is greater than seven.
 - ▶ E: Necessarily, the number of planets is greater than seven.
- Uh-oh.



Ruth Barcan Marcus and Saul Kripke wish you...



**...a necessarily
nice weekend!**