Philosophy 240
Symbolic Logic

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Class #7: Adequacy

Four Possible Unary Functions

	α
1	1
1	0

	α
1	1
0	0

~	α
0	1
1	0

	α
0	1
0	0

- We can generate each of the four possibilities with our connectives:
  - α v ~α
  - α
  - ~α
  - α • ~α

Sixteen Binary Operators

α	β		∨		⊃				≡				•				
1	**1**	1	1	1	1	0	1	1	1	0	0	0	1	0	0	0	0
1	**0**	1	1	1	0	1	1	0	0	1	1	0	0	1	0	0	0
0	**1**	1	1	0	1	1	0	1	0	1	0	1	0	0	1	0	0
0	**0**	1	0	1	1	1	0	0	1	0	1	1	0	0	0	1	0

- Our language will be expressively complete if we can generate each of the sixteen possibilities.
- A neat challenge!

Theorem 1: The Biconditional is Superfluous

- Call a connective *superfluous* if it can be defined in terms of other connectives.
- Proof 1: We can show that '$\alpha \equiv \beta$' and '$(\alpha \supset \beta) \bullet (\beta \supset \alpha)$' are logically equivalent.

Two Notions of Logical Equivalence

- LE_1: Two statements are logically equivalent iff they have the same values in every row of the truth table.
  - Semantic
- LE_2: Two statements are logically equivalent iff each is derivable from the other.
  - Proof-theoretic
  - We will start derivations next Wednesday.
- We hope that LE_1 and LE_2 yield the same results.
- To prove that LE_1 and LE_2 yield the same results, we have to justify our system of deduction.
- That work is left for another occasion.

Theorem 1: The Biconditional is Superfluous

- Proof 1: We can show that ‘α ≡ β’ and ‘(α ⊃ β) • (β ⊃ α)’ are logically equivalent.

α	≡	β
1	**1**	1
1	**0**	0
0	**0**	1
0	**1**	0

(α	⊃	β)	•	(β	⊃	α)
1	1	1	**1**	1	1	1
1	0	0	**0**	0	1	1
0	1	1	**0**	1	0	0
0	1	0	**1**	0	1	0

Theorem 2: The Conditional is Superfluous

Proof 2: Again by method of truth tables

α	⊃	β
1	**1**	1
1	**0**	0
0	**1**	1
0	**1**	0

~	α	∨	β
0	1	**1**	1
0	1	**0**	0
1	0	**1**	1
1	0	**1**	0

Combining Theorems 1 and 2

- Any sentence which can be written as a biconditional can be written in terms of negation, conjunction, and disjunction.
- Consider: ‘Dogs bite if and only if they are startled’.
  - B ≡ S
  - (B ⊃ S) • (S ⊃ B)
  - (~B ∨ S) • (~S ∨ B)

Some Questions

Q1. How can we be sure that all sentences can be written with just the five connectives?

Q2a. Can we get rid of more connectives?

Q2b. If so, what is the least number of connectives that we need?

Adequacy

- A set of connectives is called **adequate** iff corresponding to every possible truth table there is at least one sentence using only those connectives.
  - A language is *expressively complete* iff its connectives are adequate
- By “every possible truth table,” I mean every combination of 1s and 0s in the column under the main connective.

Theorem 3: {Negation, Conjunction} is adequate for languages with only one propositional variable.

Proof by sheer force

- There are only four possible truth tables:
  - 11
  - 10
  - 01
  - 00

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~	(α	•	~	α)
1	1	0	0	1
1	0	0	1	0

α
1
0

~	α
0	1
1	0

α	•	~	α
1	**0**	0	1
0	**0**	1	0

Our Goal: A General Adequacy Result

- We want to demonstrate the general theorem that the five connectives are adequate for any number of propositional variables.
- By Theorems 1 and 2, we know that the five connectives are adequate if, and only if, the three (negation, conjunction, and disjunction) are adequate.
  - **Theorem 1: The Biconditional is Superfluous**
  - **Theorem 2: The Conditional is Superfluous**

Disjunctive Normal Form (DNF)

- In order to prove our general adequacy results, we need the notion of DNF.
- A sentence is in DNF iff if is a series of disjunctions, each disjunct of which is a conjunction of simple letters or negations of simple letters.
- Sentences in DNF may not be wffs, since we will allow series of disjuncts (and series of conjuncts among the disjuncts)
- A single letter or its negation can be considered a degenerate conjunction or disjunction.
- All statements in DNF use only negations, conjunctions, and disjunctions.

DNF

- DNF

  ~P ∨ ~Q ∨ (~P • ~Q)
  (P • Q) ∨ (~P • Q) ∨ (~P • Q • ~R)
  P • ~Q • S
  ~P ∨ Q ∨ T
  P

- NDNF

  ~(P • Q)
  P ⊃ Q
  (P • ~Q) ∨ (~P ≡ Q)
  (P ∨ Q) • (~P ∨ ~Q)
  P ∨ ~Q ∨ ~(P ∨ Q)

Exercise

Which of the following sentences are in DNF?

1. (P • ~Q) ∨ (P • Q)
2. (P • Q • R) ∨ (~P • ~Q • ~R)
3. ~P ∨ Q ∨ R
4. (P ∨ Q) • (P ∨ ~R)
5. (P • Q) ∨ (P • ~Q) ∨ (~P • Q) ∨ (~P • ~R)
6. (~P • Q) • (P • R) ∨ (Q • ~R)
7. (P • ~Q • R) ∨ (Q • ~R) ∨ ~Q
8. ~(P • Q) ∨ (P • R)
9. P • Q
10. ~P

Exercise

Which of the following sentences are in DNF?

1. (P • ~Q) ∨ (P • Q)
2. (P • Q • R) ∨ (~P • ~Q • ~R)
3. ~P ∨ Q ∨ R

Not DNF 4. (P ∨ Q) • (P ∨ ~R)

5. (P • Q) ∨ (P • ~Q) ∨ (~P • Q) ∨ (~P • ~R)

Not DNF 6. (~P • Q) • (P • R) ∨ (Q • ~R)

7. (P • ~Q • R) ∨ (Q • ~R) ∨ ~Q

Not DNF 8. ~(P • Q) ∨ (P • R)

9. P • Q
10. ~P

Theorem 4: The Set of Negation, Conjunction, and Disjunction {~, •, ∨} is Adequate. (1/4)

Proof: By cases.

- For any size truth table, with any number of connectives, there are three possibilities for the column under the main connective.
- Case 1: Every row is false.
- Case 2: There is one row which is true, and every other row is false.
- Case 3: There is more than one row which is true.

Theorem 4: The Set of Negation, Conjunction, and Disjunction {~, •, ∨} is Adequate. (2/4)

Case 1: Every row is false.

- Construct a sentence with one variable in the sentence conjoined with its negation and each of the remaining variables.
- So, if you have variables P, Q, R, S, and T, you would write:
  - (P • ~P) • (Q • S • T)
- If you have more variables, add more conjuncts.
- The resulting formula, in DNF, is false in every row, and uses only conjunction and negation.

Theorem 4: The Set of Negation, Conjunction, and Disjunction {~, •, ∨} is Adequate. (3/4)

Case 2: There is one row which is true, and every other row is false.

- Consider the row in which the statement is true.
- Write a conjunction of the following statements:
  - For each variable, if it is true in that row, write that variable.
  - For each variable, if it is false in that row, write the negation of that variable.
- The resulting formula is in DNF and is true only in the prescribed row.
- Example:
  - Consider a formula with two variables: P=1100; Q=1010;
  - Main connective=0010
  - Use: ~P • Q
  - Multiple formulas will yield the same truth table: ~(P ∨ ~Q)
    - We need only one.

P	Q	Main operator
1	1	0
1	0	0
0	1	1
0	0	0

Theorem 4: The Set of Negation, Conjunction, and Disjunction {~, •, ∨} is Adequate. (4/4)

Case 3: There is more than one row which is true.

- For each row in which the statement is true, perform the method from Case 2.
- Then, form the disjunction of all the resulting formulas.
- Example:
  - Consider a formula with three variables.
  - P=11110000; Q=11001100; R=10101010
  - Main connective=10010000
  - Use: (P • Q • R) ∨ (P • ~Q • ~R)
- QED

P	Q	R	Main operator
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0

Theorem 5: The Set {∨, ~} is Adequate.

- By Theorem 4, we can write a formula for any truth table using as connectives only those in the set {∨, •, ~}.
- ‘α • β’ is equivalent to ‘~(~α ∨ ~β)’.
- So, we can replace any occurrence of ‘•’ in any formula, according to the above equivalence.
- QED
- For example, consider the sample formula from Case 3 of the proof of Theorem 4:
  - (P • Q • R) ∨ (P • ~Q • ~R)
  - [P • (Q • R)] ∨ [P • (~Q • ~R)]
  - [P • ~(~Q ∨ ~R)] ∨ [P • ~(~~Q ∨ ~~R)]
  - ~[~P ∨ ~~(~Q ∨ ~R)] ∨ ~[~P ∨ ~~(~~Q ∨ ~~R)]
  - ~[~P ∨ (~Q ∨ ~R)] ∨ ~[~P ∨ (Q ∨ R)]

Two Theorems for You to Prove

Given Theorem 4: The Set of Negation, Conjunction, and Disjunction {~, •, ∨} is Adequate.

- **Theorem 6: The set {•, ~} is adequate.**
- **Theorem 7: The set {~, ⊃} is adequate.**
- We'll use them a bit, now, even though we haven't proved them.

Theorem 8: The Set {⊃, ∨} is *Inadequate*

- To show that a set of connectives is inadequate, we can show that there is some truth table that can not be constructed using those connectives.
- Recall that both ‘$\alpha \supset \beta$’ and ‘$\alpha \vee \beta$’ are true when α and β are both true.
- Thus, using these connectives we can never construct a truth table with a false first row.
- QED

Theorem 10: The Set {~} is Inadequate.

- The only possible truth tables with one variable and ~ are 10 and 01.
- Thus, we can not generate 11 or 00.
- QED

The Sheffer Stroke

alternative denial, or not-both

α	|	β
1	0	1
1	1	0
0	1	1
0	1	0

Theorem 11: The Set {|} is Adequate.

- '~α' is logically equivalent to 'α | α'.
- 'α • β' is logically equivalent to '(α | β) | (α | β)'.
- By Theorem 6, {~, •} is adequate.
- QED

α	|	β
1	0	1
1	1	0
0	1	1
0	1	0

The Peirce Arrow

joint denial, or neither-nor

α	↓	β
1	0	1
1	0	0
0	0	1
0	1	0

Theorem 12: The Set {↓} is Adequate.

- ‘~α’ is equivalent to ‘α ↓ α’.
- ‘α ∨ β’ is equivalent to ‘(α ↓ β) ↓ (α ↓ β)’.
- Theorem 5: The set {∨, ~} is adequate.
- QED

α	↓	β
1	0	1
1	0	0
0	0	1
0	1	0

Theorem 13: ↓ and | are the Only Connectives Adequate by Themselves.

- Imagine we had another adequate connective, #.
- We know the first rows must be false, by the reasoning in Proof 8.
- Similar reasoning fills in the last row.
- Thus, '~α' is equivalent to 'α # α'.
- Now, we need to fill in the other rows.
  - If the remaining two rows are 11, then we have '|'.
  - If the remaining two rows are 00, then we have '↓'.
  - So, the only other possibilities are 10 and 01.
  - 01 yields 0011, which is just '~α'.
  - 10 yields 0101, which is just '~β'.
  - By Theorem 10, {~} is inadequate.
- QED

α	#	β
1	0	1
1		0
0		1
0	1	0

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