Philosophy 240
Symbolic Logic

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Class #34
More Translation Using Relational Predicates
Rules of Passage

Quantifiers and Relational Predicates

Bxy: x is bigger than y

- Joe is bigger than some thing.
  - $(\exists x)Bjx$
- Something is bigger than Joe.
  - $(\exists x)Bxj$
- Joe is bigger than everything.
  - $(\forall x)Bjx$
- Everything is bigger than Joe.
  - $(\forall x)Bxj$

Overlapping Quantifiers

Lxy: x loves y

- Everything loves something.
  - $(\forall x)(\exists y)Lxy$
- Something loves everything.
  - $(\exists x)(\forall y)Lxy$
- $(\forall x)(\exists y)Lyx$
  - Everything is loved by something.
- $(\exists x)(\forall y)Lyx$
  - Something is loved by everything.

Our Original Argument

Finally Translated

Consider:

1. Bob is taller than Charles.
2. Andrew is taller than Bob.
3. For any x, y and z, if x is taller than y and y is taller than z, then x is taller than z.

So, Andrew is taller than Charles.

1. Tbc
2. Tab
3. $(\forall x)(\forall y)(\forall z)[(Txy \bullet Tyz) \supset Txz]$ / Tac

Derivation on Wednesday

More Examples

- Something teaches Plato. (Txy: x teaches y)
  - $(\exists x)Txp$
- Someone teaches Plato. (Px: x is a person)
  - $(\exists x)(Px \bullet Txp)$
- Plato teaches everyone.
  - $(\forall x)(Px \supset Tpx)$
- Everyone teaches something.
  - $(\forall x)[Px \supset (\exists y)Txy]$
- Some people teach themselves.
  - $(\exists x)(Px \bullet Txx)$
- There are teachers.
  - $(\exists x)(\exists y)Txy$
- There are students.
  - $(\exists x)(\exists y)Tyx$
- Skilled teachers are interesting.
  - $(\forall x)[(\exists y)Txy \supset (Sx \supset Ix)]$
- Skilled teachers are better than unskilled teachers.
  - $(\forall x)\{[(\exists y)Txy \bullet Sx] \supset \{(\forall z)[(\exists w)Tzw \bullet {\sim}Sz] \supset Bxz\}\}$

Wide and Narrow Scope

- Wide
  - $(\exists x)(\exists y)[(Px \bullet Py) \bullet Lxy]$
  - $(\forall x)(\forall y)[Px \supset (Qy \supset Rxy)]$
- Narrow
  - $(\exists x)[Px \bullet (\exists y)(Py \bullet Lxy)]$
  - $(\forall x)[Px \supset (\forall y)(Qy \supset Rxy)]$
- It's generally better to give your quantifiers as narrow a scope as possible.
- Not equivalent:
  - $(\forall x)[Px \supset (\exists y)(Py \bullet Qxy)]$
    - 'all people love someone'
  - $(\exists y)(\forall x)[Px \supset (Py \bullet Qxy)]$
    - 'there is someone everyone loves'

Moving Quantifiers

Of the same type

- In some cases, we can move quantifiers around without much worry.
  - If quantifiers are of the same type, we can push them in or pull them out.
  - Be careful not to accidentally bind any variables!
- Everyone loves everyone
  - $(\forall x)[Px \supset (\forall y)(Py \supset Lxy)]$
  - $(\forall x)(\forall y)[(Px \cdot Py) \supset Lxy]$
  - $(\forall y)(\forall x)[(Px \cdot Py) \supset Lxy]$
- Someone loves someone
  - $(\exists x)[Px \cdot (\exists y)(Py \cdot Lxy)]$
  - $(\exists x)(\exists y)[(Px \cdot Py) \cdot Lxy]$
  - $(\exists y)(\exists x)[(Px \cdot Py) \cdot Lxy]$

Mixing Quantifiers

- None of the following examples are equivalent:
  - Everyone loves someone: $(\forall x)(\exists y)[Px \supset (Py \cdot Lxy)]$
  - Everyone is loved by someone: $(\forall x)(\exists y)[Px \supset (Py \cdot Lyx)]$
  - Someone loves everyone: $(\exists x)(\forall y)[Px \cdot (Py \supset Lxy)]$
  - Someone is loved by everyone: $(\exists x)(\forall y)[Px \cdot (Py \supset Lyx)]$
- The first word in each translation above corresponds to the leading quantifier.
- The connectives which directly follow the ‘Px’ and the ‘Py’ are determined by the quantifier binding that variable.

Using Narrow Scope

- Everyone loves someone.
  $(\forall x)[Px \supset (\exists y)(Py \cdot Lxy)]$
- Everyone is loved by someone.
  $(\forall x)[Px \supset (\exists y)(Py \bullet Lyx)]$
- Someone loves everyone.
  $(\exists x)[Px \cdot (\forall y)(Py \supset Lxy)]$
- Someone is loved by everyone.
  $(\exists x)[Px \cdot (\forall y)(Py \supset Lyx)]$

Moving Mixed Quantifiers: A Problem

- The following sentences are *not* equivalent
  - $(\forall x)[(\exists y)Lxy \supset Hx]$
    For any x, if there is a y that x loves, then x is happy.
    All lovers are happy.
  - $(\forall x)(\exists y)(Lxy \supset Hx)$
    For any x, there is a y such that if x loves y then x is happy.
- The first does not commit to the existence of something that, by being loved, makes a person happy.
- The second does.

Prenex Normal Form (PNF)

- Some metalogical proofs require all statements of **F** to be written with all quantifiers having wide scope.
- A sentence is in Prenex Normal Form (PNF) if all of its quantifiers are in the front, having wide scope.
- Rules of Passage allow us to transform all statements of **F** into PNF.
- They are rules of replacement.
- I will not require that you use them in proofs.
- They may be useful in learning how to translate.

Rules of Passage

- For all variables α and all formulas Γ and Δ:

  RP1: $(\exists\alpha)(\Gamma \vee \Delta) \rightleftarrows (\exists\alpha)\Gamma \vee (\exists\alpha)\Delta$

  RP2: $(\forall\alpha)(\Gamma \bullet \Delta) \rightleftarrows (\forall\alpha)\Gamma \bullet (\forall\alpha)\Delta$

- For all variables α, all formulas Γ containing α, and all formulas Δ not containing α:

  RP3: $(\exists\alpha)(\Delta \bullet \Gamma\alpha) \rightleftarrows \Delta \bullet (\exists\alpha)\Gamma\alpha$

  RP4: $(\forall\alpha)(\Delta \bullet \Gamma\alpha) \rightleftarrows \Delta \bullet (\forall\alpha)\Gamma\alpha$

  RP5: $(\exists\alpha)(\Delta \vee \Gamma\alpha) \rightleftarrows \Delta \vee (\exists\alpha)\Gamma\alpha$

  RP6: $(\forall\alpha)(\Delta \vee \Gamma\alpha) \rightleftarrows \Delta \vee (\forall\alpha)\Gamma\alpha$

  RP7: $(\exists\alpha)(\Delta \supset \Gamma\alpha) \rightleftarrows \Delta \supset (\exists\alpha)\Gamma\alpha$

  RP8: $(\forall\alpha)(\Delta \supset \Gamma\alpha) \rightleftarrows \Delta \supset (\forall\alpha)\Gamma\alpha$

  RP9: $(\exists\alpha)(\Gamma\alpha \supset \Delta) \rightleftarrows (\forall\alpha)\Gamma\alpha \supset \Delta$

  RP10: $(\forall\alpha)(\Gamma\alpha \supset \Delta) \rightleftarrows (\exists\alpha)\Gamma\alpha \supset \Delta$

- These are rules of replacement and may be used on parts of lines.

(Slightly) Friendlier (Maybe) Versions of the Rules of Passage

RP1: $(\exists x)(Px \vee Qx) \rightleftarrows (\exists x)Px \vee (\exists x)Qx$

RP2: $(\forall x)(Px \bullet Qx) \rightleftarrows (\forall x)Px \bullet (\forall x)Qx$

RP3: $(\exists x)(\mathscr{F} \bullet Px) \rightleftarrows \mathscr{F} \bullet (\exists x)Px$

RP4: $(\forall x)(\mathscr{F} \bullet Px) \rightleftarrows \mathscr{F} \bullet (\forall x)Px$

RP5: $(\exists x)(\mathscr{F} \vee Px) \rightleftarrows \mathscr{F} \vee (\exists x)Px$

RP6: $(\forall x)(\mathscr{F} \vee Px) \rightleftarrows \mathscr{F} \vee (\forall x)Px$

RP7: $(\exists x)(\mathscr{F} \supset Px) \rightleftarrows \mathscr{F} \supset (\exists x)Px$

RP8: $(\forall x)(\mathscr{F} \supset Px) \rightleftarrows \mathscr{F} \supset (\forall x)Px$

RP9: $(\exists x)(Px \supset \mathscr{F}) \rightleftarrows (\forall x)Px \supset \mathscr{F}$

RP10: $(\forall x)(Px \supset \mathscr{F}) \rightleftarrows (\exists x)Px \supset \mathscr{F}$

Examples

RP4: $(\forall\alpha)(\Delta \bullet \Gamma\alpha) \rightleftarrows \Delta \bullet (\forall\alpha)\Gamma\alpha$
RP8: $(\forall\alpha)(\Delta \supset \Gamma\alpha) \rightleftarrows \Delta \supset (\forall\alpha)\Gamma\alpha$
RP9: $(\exists\alpha)(\Gamma\alpha \supset \Delta) \rightleftarrows (\forall\alpha)\Gamma\alpha \supset \Delta$
RP10: $(\forall\alpha)(\Gamma\alpha \supset \Delta) \rightleftarrows (\exists\alpha)\Gamma\alpha \supset \Delta$

- Using RP4:
  - $(\exists x)[Px \bullet (\forall y)(Qy \supset Rxy)]$
  - $(\exists x)(\forall y)[Px \bullet (Qy \supset Rxy)]$
- Using RP8:
  - $(\forall x)(\forall y)[Px \supset (Qy \supset Rxy)]$
  - $(\forall x)[Px \supset (\forall y)(Qy \supset Rxy)]$
- Using RP9:
  - $(\forall x)(\exists y)(Lxy \supset Hx)$
  - $(\forall x)[(\forall y)Lxy \supset Hx]$
- Using RP10:
  - $(\forall x)[(\exists y)Lxy \supset Hx]$
  - $(\forall x)(\forall y)(Lxy \supset Hx)$
- Also Using RP10:
  - $(\forall x)[Px \supset (\exists y)Qy]$
  - $(\exists x)Px \supset (\exists y)Qy$

Proving RP10

RP10: $(\forall\alpha)(\Gamma\alpha \supset \Delta) \rightleftarrows (\exists\alpha)\Gamma\alpha \supset \Delta$

- Consider first what happens when Δ is true, and then when Δ is false.
- If Δ is true, then both formulas will turn out to be true.
  - The consequent of the formula on the right is just Δ.
  - So, if Δ is true, the whole formula on the right will be true.
  - On the left, Γα ⊃ Δ will be true for every instance of α, since the consequent is true.
  - So, the universal generalization of each such formula will be true.
- If Δ is false, then the truth value of each formula will depend.
  - To show that the truth values of each formula will be the same, we will show that the formula on the right is true in every case that the formula on the left is true and that the formula on the left is true in every case that the formula on the right is true.
  - If the formula on the left turns out to be true when Δ is false, it must be because Γα is false, for every α.
  - But then, (∃α)Γα will be false, and so the formula on the right turns out to be true.
  - If the formula on the right turns out to be true, then it must be because (∃α)Γα is false.
  - And so, there will be no value of α that makes Γα true, and so the formula on the right will also turn out to be (vacuously) true.
- QED

Rules of Passage in Translations

RP7: $(\exists\alpha)(\Delta \supset \Gamma\alpha) \rightleftarrows \Delta \supset (\exists\alpha)\Gamma\alpha$
RP8: $(\forall\alpha)(\Delta \supset \Gamma\alpha) \rightleftarrows \Delta \supset (\forall\alpha)\Gamma\alpha$
RP9: $(\exists\alpha)(\Gamma\alpha \supset \Delta) \rightleftarrows (\forall\alpha)\Gamma\alpha \supset \Delta$
RP10: $(\forall\alpha)(\Gamma\alpha \supset \Delta) \rightleftarrows (\exists\alpha)\Gamma\alpha \supset \Delta$

- If anything was damaged, then everyone gets upset.
  - $(\exists x)Dx \supset (\forall x)(Px \supset Ux)$
  - $(\forall x)[Dx \supset (\forall y)(Py \supset Uy)]$ by RP10
- If there are any wildebeest, then if all lions are hungry, they will be eaten.
  - $(\forall x)\{Wx \supset [(\forall y)(Ly \supset Hy) \supset Ex]\}$
  - $(\forall x)\{Wx \supset (\exists y)[(Ly \supset Hy) \supset Ex]\}$ by RP9
  - $(\forall x)(\exists y)\{Wx \supset [(Ly \supset Hy) \supset Ex]\}$ by RP7

Prenex Normal Form

RP4: $(\forall\alpha)(\Delta \bullet \Gamma\alpha) \rightleftarrows \Delta \bullet (\forall\alpha)\Gamma\alpha$
RP7: $(\exists\alpha)(\Delta \supset \Gamma\alpha) \rightleftarrows \Delta \supset (\exists\alpha)\Gamma\alpha$
RP9: $(\exists\alpha)(\Gamma\alpha \supset \Delta) \rightleftarrows (\forall\alpha)\Gamma\alpha \supset \Delta$
RP10: $(\forall\alpha)(\Gamma\alpha \supset \Delta) \rightleftarrows (\exists\alpha)\Gamma\alpha \supset \Delta$

- Sentences do not have unique PNFs.
- If there is a philosopher whom all philosophers contradict, then there is a philosopher who contradicts him or herself.
  - $(\exists x)[Fx \bullet (\forall y)(Fy \supset Gyx)] \supset (\exists z)(Fz \bullet Gzz)$
- Transformation #1
  - $(\exists z)\{(\exists x)[Fx \bullet (\forall y)(Fy \supset Gyx)] \supset (Fz \bullet Gzz)\}$ by RP7
  - $(\exists z)\{(\exists x)(\forall y)[Fx \bullet (Fy \supset Gyx)] \supset (Fz \bullet Gzz)\}$ by RP4
  - $(\exists z)(\forall x)\{(\forall y)[Fx \bullet (Fy \supset Gyx)] \supset (Fz \bullet Gzz)\}$ by RP10
  - $(\exists z)(\forall x)(\exists y)\{[Fx \bullet (Fy \supset Gyx)] \supset (Fz \bullet Gzz)\}$ by RP9
- Transformation #2
  - $(\forall x)\{[Fx \bullet (\forall y)(Fy \supset Gyx)] \supset (\exists z)(Fz \bullet Gzz)\}$ by RP10
  - $(\forall x)\{(\forall y)[Fx \bullet (Fy \supset Gyx)] \supset (\exists z)(Fz \bullet Gzz)\}$ by RP4
  - $(\forall x)(\exists y)\{[Fx \bullet (Fy \supset Gyx)] \supset (\exists z)(Fz \bullet Gzz)\}$ by RP9
  - $(\forall x)(\exists y)(\exists z)\{[Fx \bullet (Fy \supset Gyx)] \supset (Fz \bullet Gzz)\}$ by RP7
- The results are in prenex form, and logically equivalent to the original sentence.
- But, they differ in form from each other.

More Translations

1. Everyone loves something. (Px, Lxy: x loves y)
   - $(\forall x)[Px \supset (\exists y)Lxy]$
2. No one knows everything. (Px, Kxy: x knows y)
   - $(\forall x)[Px \supset (\exists y){\sim}Kxy]$
3. No one knows everyone.
   - $(\forall x)[Px \supset (\exists y)(Py \bullet {\sim}Kxy)]$
4. Every woman is stronger than some man. (Wx, Mx, Sxy: x is stronger than y)
   - $(\forall x)[Wx \supset (\exists x)(Mx \bullet Sxy)]$
5. No cat is smarter than any horse. (Cx, Hx, Sxy: x is smarter than y)
   - $(\forall x)[Cx \supset {\sim}(\exists y)(Hy \bullet Sxy)]$
   - $(\forall x)[Cx \supset (\forall y)(Hy \supset {\sim}Sxy)]$

Even More Translations

6. Dead men tell no tales. (Dx, Mx, Tx, Txy: x tells y)
- $(\forall x)[(Dx \bullet Mx) \supset (\forall y)(Ty \supset \sim Txy)]$

7. There is a city between New York and Washington. (Cx, Bxyz: y is between x and z)
- $(\exists x)(Cx \bullet Bnxw)$

8. Everyone gives something to someone. (Px, Gxyz: y gives x to z)
- $(\forall x)[Px \supset (\exists y)(\exists z)(Pz \bullet Gyxz)]$

9. A dead lion is more dangerous than a live dog. (Ax: x is alive, Lx, Dx, Dxy: x is more dangerous than y)
- $(\forall x)\{(Lx \bullet \sim Ax) \supset (\forall y)[(Dy \bullet Ay) \supset Dxy]\}$

10. A lawyer who pleads his own case has a fool for a client. (Lx, Fx, Pxy: x pleads y's case; Cxy: y is a client of x)
- $(\forall x)[(Lx \bullet Pxx) \supset (\exists y)(Fy \bullet Cxy)]$
- $(\forall x)[(Lx \bullet Pxx) \supset Fx]$