Philosophy 240
Symbolic Logic

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Class #27: Modal Logic


BOO

Sentential Operators

- In **PL** (and **M**), we have four binary sentential operators.
- We also have a unary sentential operator: negation.
- There are other ways to modify sentences.
- Negation
  - It is not the case that the sun is shining.
- Modal Operators
  - It is possible that the sun is shining.
  - It is necessary that the sun is shining.

Propositional Modal Logic (PML)

Formation Rules

1. A single capital English letter is a wff.
2. If α is a wff, so is ~α.
3. If α and β are wffs, then so are (α · β), (α ∨ β), (α ⊃ β), and (α ≡ β).
4. If α is a wff, then so are ◊α and □α.
5. These are the only ways to make wffs.

The Relation between ◊ and □

Compare to Quantifier Equivalence

- $\Box\alpha \rightleftarrows \sim\Diamond\sim\alpha$
- $\Diamond\alpha \rightleftarrows \sim\Box\sim\alpha$

Interpretations of ◊ and □

- Alethic
  - ‘It is possible that’ and ‘It is necessary that’
- Deontic
  - ‘It is morally permissible that’ and ‘it is morally required that’
- Temporal
  - ‘At some time in the future’ and ‘at all times in the future’
- Dynamic
  - ‘after some change’ and ‘after any change’
- Epistemic
  - Epistemic: □ is ‘it is known that’
  - Doxastic: □ is ‘it is believed that’

Actual World Semantics

$\mathcal{V}(\sim\alpha) = 1$ if $\mathcal{V}(\alpha) = 0$; otherwise $\mathcal{V}(\sim\alpha) = 0$

$\mathcal{V}(\alpha \bullet \beta) = 1$ if $\mathcal{V}(\alpha) = 1$ and $\mathcal{V}(\beta) = 1$; otherwise $\mathcal{V}(\alpha \bullet \beta) = 0$

$\mathcal{V}(\alpha \vee \beta) = 1$ if $\mathcal{V}(\alpha) = 1$ or $\mathcal{V}(\beta) = 1$; otherwise $\mathcal{V}(\alpha \bullet \beta) = 0$

$\mathcal{V}(\alpha \supset \beta) = 1$ if $\mathcal{V}(\alpha) = 0$ or $\mathcal{V}(\beta) = 1$; otherwise $\mathcal{V}(\alpha \supset \beta) = 0$

$\mathcal{V}(\alpha \equiv \beta) = 1$ if $\mathcal{V}(\alpha) = \mathcal{V}(\beta)$; otherwise $\mathcal{V}(\alpha \equiv \beta) = 0$

Possible World Semantics (Leibnizian)

- $\mathcal{V}(\Box\alpha) = 1$ if $\mathcal{V}(\alpha, w_n) = 1$ for all w_n in $\mathcal{U}$
- $\mathcal{V}(\Box\alpha) = 0$ if $\mathcal{V}(\alpha, w_n) = 0$ for any w_n in $\mathcal{U}$
- $\mathcal{V}(\Diamond\alpha) = 1$ if $\mathcal{V}(\alpha, w_n) = 1$ for any w_n in $\mathcal{U}$
- $\mathcal{V}(\Diamond\alpha) = 0$ if $\mathcal{V}(\alpha, w_n) = 0$ for all w_n in $\mathcal{U}$

$\mathcal{U} = \{w_1, w_2, w_3\}$

At w_1, P, Q, R and S are all true.
At w_2, P and Q are true, but R and S are false.
At w_3, P is true, and Q, R, and S are false.

1. $\Box(P \supset Q)$
2. $\Diamond(P \supset Q)$
3. $\Box P \supset \Box Q$
4. $\Diamond P \supset \Diamond Q$
5. $\Diamond[(Q \vee {\sim}R) \supset {\sim}P]$
6. $\Diamond P \supset [Q \supset \Box(R \bullet S)]$

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$\mathcal{U} = \{w_1, w_2, w_3\}$

At w_1, P, Q, R and S are all true.

At w_2, P and Q are true, but R and S are false.

At w_3, P is true, and Q, R, and S are false.

False	1. $\Box(P \supset Q)$
True	2. $\Diamond(P \supset Q)$
False	3. $\Box P \supset \Box Q$
True	4. $\Diamond P \supset \Diamond Q$
False	5. $\Diamond[(Q \vee {\sim}R) \supset {\sim}P]$
True at w_3, False otherwise	6. $\Diamond P \supset [Q \supset \Box(R \bullet S)]$

Different Kinds of Possibility

- Leibnizian semantics is the modal logic of logical possibility.
  - (Called **S5** by C.I. Lewis)
- Logical possibility is very weak.
  - It is logically possible for a bachelor to be married,
  - ...for objects to travel faster than the speed of light,
  - ...for a square to have five sides.
- Semantic possibility
- Physically possibility
- Mathematical possibility

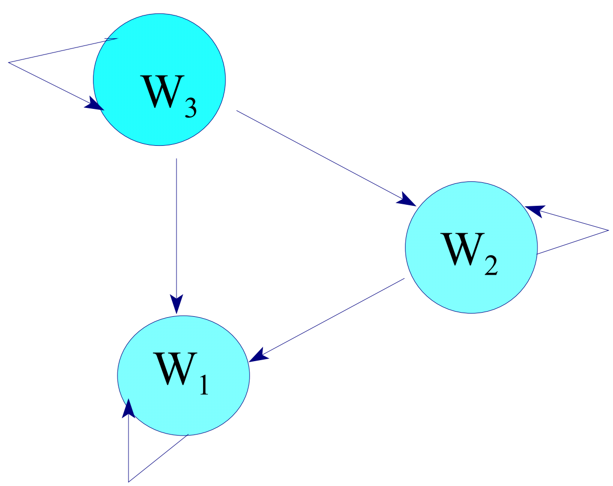
Worlds with Different Possibilities

- Possibility might vary among worlds.
- World One: ours.
- World Two: like ours, except that there is some force which moves the planets in perfectly circular orbits.
- The law that says that all planets move in elliptical orbits holds in both worlds, since a circle is just a type of ellipse.
- w_2 obeys all the laws of w_1.
- w_1 does not obey all the laws of w_2.

Possible World Semantics (Kripkean)

Adds Accessibility Relations

$\mathcal{R} = \{<w_1, w_1>, <w_2, w_1>, <w_2, w_2>, <w_3, w_1>, <w_3, w_2>, <w_3, w_3>\}$



Possible World Semantics (Kripkean)

Valuation Rules

- $\mathcal{V}(\Box\alpha, w_n) = 1$ if $\mathcal{V}(\alpha, w_m) = 1$ for all w_m in $\mathcal{U}$ such that $\langle w_n, w_m \rangle$ is in $\mathcal{R}$
- $\mathcal{V}(\Box\alpha, w_n) = 0$ if $\mathcal{V}(\alpha, w_m) = 0$ for any w_m in $\mathcal{U}$ such that $\langle w_n, w_m \rangle$ is in $\mathcal{R}$
- $\mathcal{V}(\Diamond\alpha, w_n) = 1$ if $\mathcal{V}(\alpha, w_m) = 1$ for any w_m in $\mathcal{U}$ such that $\langle w_n, w_m \rangle$ is in $\mathcal{R}$
- $\mathcal{V}(\Diamond\alpha, w_n) = 0$ if $\mathcal{V}(\alpha, w_m) = 0$ for all w_m in $\mathcal{U}$ such that $\langle w_n, w_m \rangle$ is in $\mathcal{R}$

1. $\Box(P \supset Q)_1$
2. $\Box(P \supset Q)_3$
3. $\Diamond{\sim}(Q \lor R)_1$
4. $\Diamond{\sim}(Q \lor R)_2$
5. $\Diamond{\sim}(Q \lor R)_3$
6. $\Box P_1 \supset \Box Q_1$
7. $\Box P_3 \supset \Box Q_3$

Systems of Inference

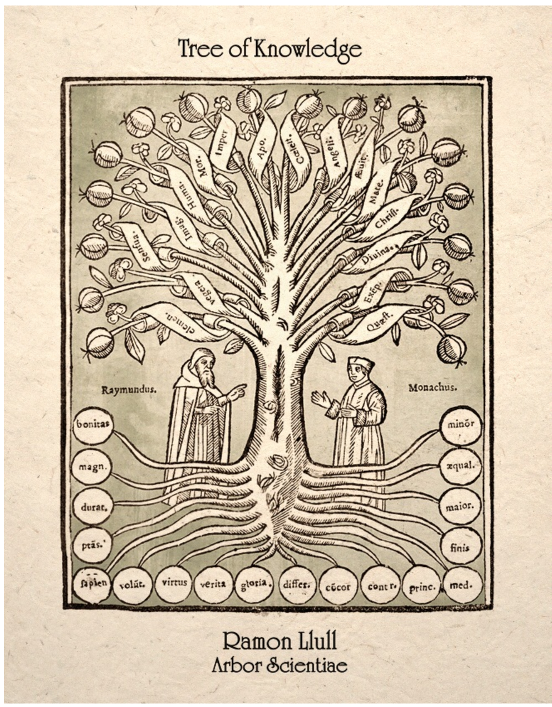
S5 Axioms and Rules

▪ PL	Any tautology of **PL**
▪ Duality	$\Diamond\alpha \equiv \sim\Box\sim\alpha$
▪ K	$\Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)$
▪ T	$\Box\alpha \supset \alpha$
▪ 5	$\Diamond\alpha \supset \Box\Diamond\alpha$
▪ Necessitation	$\alpha \vdash \Box\alpha$
▪ MP	$\alpha \supset \beta, \alpha \vdash \beta$

Other Modal Systems

- In **S5**, every world is accessible from every other world.
- Weaker modal logics have weaker accessibility relations and characteristic axioms.
  - The axioms and the accessibility relations go together.
- **K** is weak
  - $\Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)$
- **D** is stronger
  - $\Box\alpha \supset \Diamond\alpha$
  - Can't derive T ($\Box\alpha \supset \alpha$) in **D**
  - So **D is** not apt for the alethic interpretation of the operators.
  - But the deontic interpretation of the characteristic axiom of **D** looks better.

Hintikka's Epistemic Logic



- K $\Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)$
- T $\Box\alpha \supset \alpha$
- 4 $\Box\alpha \supset \Box\Box\alpha$
  - KK Thesis

Hintikka and the KK Thesis

- "Suppose we say that evidence for a proposition, P, is conclusive iff it is so strong that, once one discovers it, further inquiry cannot give one reason to stop believing P. The concept of knowledge used by many philosophers seems to be a strong one on which one knows P only if one's evidence for P is conclusive in this sense. It is plausible that the KK principle holds for this strong concept of knowledge. For it is plausible that one's evidence for P is conclusive in the above sense only if it rules out the possibility that one does not know P, and thus only if it allows one to know that one knows P.
- "To see this, suppose one has evidence, E, for a proposition P, and that E does not rule out the possibility that one does not know P. If E does not rule out this possibility, then, after one has discovered E, further inquiry can, in principle, reveal to one that one does not know P. But if further inquiry were to reveal this, then it would surely give one reason to stop believing P (since one should not believe things that one does not know). So it is plausible that, if E does not rule out the possibility that one does not know P, then it is not conclusive in the sense just defined, and hence plausible that, if knowledge requires evidence that is conclusive in this sense, the KK principle holds" (Hintikka 1970: 145-6).

Concerns about Possible Worlds

- Metaphysical
  - What is a possible world?
  - Do possible worlds exist?
- Epistemic
  - How do we know about possible worlds?
  - Do we stipulate them?
  - Do we discover them, or facts about them?

A Quinean Concern

- Consider:
  - A. Nine is greater than seven.
  - B: The number of planets is greater than seven.
- A and B have the same truth value.
  - One can be inferred from the other by a simple substitution, given that:
  - C: The number of planets = nine.
- Now Consider
  - D: Necessarily, nine is greater than seven.
  - E: Necessarily, the number of planets is greater than seven.
- Uh-oh.