# Philosophy 240
# *Symbolic Logic*


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Class #20: On Axiomatic Systems

# Hunter's System PS

## With axiom schemata

- Three axiom schemata:

  PS1: $\alpha \supset (\beta \supset \alpha)$

  PS2: $(\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$

  PS3: $(\sim\alpha \supset \sim\beta) \supset (\beta \supset \alpha)$

  Any formula of the language PS of any of the forms PS1, PS2, or PS3 is an axiom of PS.

  – Infinitely many axioms of PS

- One rule of inference, modus ponens.

  If α and β are formulas, β is a consequence in system PS of α and $\alpha \supset \beta$.

  $\alpha, (\alpha \supset \beta) \vdash_{PS} \beta$

# Hunter's System PS

With a substitution rule

- Three axioms
  PS1*: $P \supset (Q \supset P)$
  PS2*: $(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$
  PS3*: $(\sim P \supset \sim Q) \supset (Q \supset P)$
- Two rules of inference
  MP: $\alpha, (\alpha \supset \beta) \vdash_{PS} \beta$
  Substitution: For any wff α, and any wff β appearing as a sub-formula in wff δ: If $\vdash_{PS} \delta$, then $\vdash_{PS} \delta_{\alpha}^{\beta}$
  – where '$\delta_{\alpha}^{\beta}$' is the result of substituting 'α' for 'β' throughout.

# Hunter's PS and Our PL

- Hunter's axiomatic system is provably equivalent to our more-familiar natural deduction system PL.
  - A metalogical result (the deduction theorem) allows us to move between the two kinds of systems.
- Frege's original *Begriffsschrift* used a Hilbert-style axiomatic system, with a different axiomatization.
  - See the appendices in Richard Mendelsohn's *The Philosophy of Gottlob Frege*, for a translation of the *Begriffsschrift* into modern notation.
- Natural deduction systems like PS in *What Follows* are due largely to Gerhard Gentzen's work in the 1930s and 1940s.
  - There is a paper on the history of natural deduction by Pelletier which might be worth a look for a term paper.
- Completeness:
  - Frege's axiomatization, Hunter's PS, and standard systems of natural deduction are all complete.
    - All logical truths are provable within the system.
- One's choice of system is thus arbitrary among the various complete systems.

# A Derivation in PS

PS1: $\alpha \supset (\beta \supset \alpha)$
PS2: $(\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$
PS3: $(\sim\alpha \supset \sim\beta) \supset (\beta \supset \alpha)$

- A1: $\vdash_{PS}$ $P \supset P$
    1. $P \supset ((P \supset P) \supset P)$ PS1
    2. $(P \supset ((P \supset P) \supset P)) \supset ((P \supset (P \supset P)) \supset (P \supset P))$ PS2
    3. $(P \supset (P \supset P)) \supset (P \supset P)$ MP, 1, 2
    4. $P \supset (P \supset P)$ PS1
    5. $P \supset P$ 3, 4, MP
- QED

# A Further Derivation in PS

PS1: $\alpha \supset (\beta \supset \alpha)$
PS2: $(\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$
PS3: $(\sim\alpha \supset \sim\beta) \supset (\beta \supset \alpha)$
A1: $P \supset P$

- A2: $\vdash_{PS} \sim P \supset (P \supset P)$

| | |
|---|---|
| 1. $P \supset P$ | Ex A1 |
| 2. $(P \supset P) \supset (\sim P \supset (P \supset P))$ | PS 1 |
| 3. $\sim P \supset (P \supset P)$ | MP, 1, 2 |

- QED

# A Longer Proof in PS

- $\vdash_{PS}$ **$(P \supset (Q \supset R)) \supset (Q \supset (P \supset R))$**
  1. $Q \supset (P \supset Q)$
  2. $(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$
  3. $[(P \supset Q) \supset (P \supset R)] \supset [Q \supset ((P \supset Q) \supset (P \supset R))]$
  4. $[((P \supset Q) \supset (P \supset Q')) \supset (Q \supset ((P \supset Q) \supset (P \supset R)))] \supset \{(P \supset (Q \supset R)) \supset [((P \supset Q) \supset (P \supset R)) \supset (Q \supset ((P \supset Q) \supset (P \supset R)))]\}$
  5. $(P \supset (Q \supset R)) \supset [((P \supset Q) \supset (P \supset R)) \supset (Q \supset ((P \supset Q) \supset (P \supset R)))]$
  6. $\{(P \supset (Q \supset R)) \supset [((P \supset Q) \supset (P \supset R)) \supset (Q \supset ((P \supset Q) \supset (P \supset R)))]\} \supset \{[(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))] \supset [(P \supset (Q \supset R)) \supset (Q \supset ((P \supset Q) \supset (P \supset R)))]\}$
  7. $[(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))] \supset [(P \supset (Q \supset R)) \supset (Q \supset ((P \supset Q) \supset (P \supset R)))]$
  8. $(P \supset (Q \supset R)) \supset [Q \supset ((P \supset Q) \supset (P \supset R))]$
  9. $[Q \supset ((P \supset Q) \supset (P \supset R))] \supset [(Q \supset (P \supset Q)) \supset (Q \supset (P \supset R))]$
  10. $[(Q \supset ((P \supset Q) \supset (P \supset R))) \supset ((Q \supset (P \supset Q)) \supset (Q \supset (P \supset R)))] \supset \{(P \supset (Q \supset R)) \supset [(Q \supset ((P \supset Q) \supset (P \supset R))) \supset ((Q \supset (P \supset Q)) \supset (Q \supset (P \supset R)))]\}$
  11. $(P \supset (Q \supset R)) \supset [(Q \supset ((P \supset Q) \supset (P \supset R))) \supset ((Q \supset (P \supset Q)) \supset (Q \supset (P \supset R)))]$
  12. $\{(P \supset (Q \supset R)) \supset [(Q \supset ((P \supset Q) \supset (P \supset R))) \supset ((Q \supset (P \supset Q)) \supset (Q \supset (P \supset R)))]\} \supset \{[(P \supset (Q \supset R)) \supset (Q \supset ((P \supset Q) \supset (P \supset R)))] \supset [(P \supset (Q \supset R)) \supset ((Q \supset (P \supset Q)) \supset (Q \supset (P \supset R)))]\}$
  13. $[(P \supset (Q \supset R)) \supset (Q \supset ((P \supset Q) \supset (P \supset R)))] \supset [(P \supset (Q \supset R)) \supset ((Q \supset (P \supset Q)) \supset (Q \supset (P \supset R)))]$
  14. $(P \supset (Q \supset R)) \supset [(Q \supset (P \supset Q)) \supset (Q \supset (P \supset R))]$
  15. $\{(P \supset (Q \supset R)) \supset [(Q \supset (P \supset Q)) \supset (Q \supset (P \supset R))]\} \supset \{[(P \supset (Q \supset R)) \supset (Q \supset (P \supset Q))] \supset [(P \supset (Q \supset R)) \supset (Q \supset (P \supset R))]\}$
  16. $[(P \supset (Q \supset R)) \supset (Q \supset (P \supset Q))] \supset [(P \supset (Q \supset R)) \supset (Q \supset (P \supset R))]$
  17. $(Q \supset (P \supset Q)) \supset [(P \supset (Q \supset R)) \supset (Q \supset (P \supset Q))]$
  18. $(P \supset (Q \supset R)) \supset (Q \supset (P \supset Q))$
  19. $(P \supset (Q \supset R)) \supset (Q \supset (P \supset R))$
- QED