# Philosophy 240: Symbolic Logic

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Class #41 - Second-Order Quantification

# Second-Order Inferences

- Consider a red apple and a red fire truck.

  (∃x)(Rx • Ax)
  (∃x)(Rx • Fx)

- We might want to infer that they have something in common, that they share a property.

  1. (∃x)(Rx • Ax)
  2. (∃x)(Rx • Fx)
  3. Ra • Aa 1, EI
  4. Rb • Ab 3, EI
  5. Ra 3, Simp
  6. Rb 4, Simp
  7. Ra • Rb5, 6, Conj
  8. (∃X)(Xa • Xb) 7, by existential generalization over predicates

# Predicate Variables

- In the prior slide, I treated the predicate ‘R’ as subject to quantification, like a singular term.
- A language which allows quantification over predicate places is called a second-order language.
- A system of logic which uses a second-order language is called second-order logic.
  - We’ll call our logic **S**.
- Second-Order logic is controversial.
  - Let’s look at it first.
  - Then we can talk about the controversy.

# Vocabulary of S

- Capital letters
  - A...U, used as predicates
  - **V, W, X, Y, and Z, used as predicate variables**
- Lower case letters
  - a, b, c, d, e, i, j, k...u are used as constants.
  - f, g, and h are used as functors.
  - v, w, x, y, z are used as singular variables.
- Five connectives: ~, •, ∨, ⊃ ≡
- Quantifiers: ∃, ∀
- Punctuation: (), [], {}

# Formation Rules for Wffs of S

1. An n-place predicate or predicate variable followed by n terms (constants, variables, or functor terms) is a wff.

2. For any singular variable β, if α is a wff that does not contain either '(∃β)' or '(∀β)', then '(∃β)α' and '(∀β)α' are wffs.

3. For any predicate variable β, if α is a wff that does not contain either '(∃β)' or '(∀β)', then '(∃β)α' and '(∀β)α' are wffs.

4. If α is a wff, so is ∼α.

5. If α and β are wffs, then so are:
- (α · β)
- (α ∨ β)
- (α ⊃ β)
- (α ≡ β)

6. These are the only ways to make wffs.

# Uses of Predicate Variables

- No two distinct things have all properties in common.
  - $(\forall x)(\forall y)[x \neq y \supset (\exists X)(Xx \bullet {\sim}Xy)]$
- Identical objects share all properties (Leibniz's law).
  - $(\forall x)(\forall y)[x = y \supset (\forall Y)(Yx \equiv Yy)]$
- The identity of indiscernibles
  - $(\forall x)(\forall y)[(\forall Z)(Zx \equiv Zy) \supset x = y]$
- The Law of the Excluded Middle
  - $(\forall X)(X \vee {\sim}X)$
- Analogies: Cat is to meow as dog is to bark.
  - $(\exists X)(Xcm \bullet Xdb)$
- The first-order mathematical induction schema can be written as a single axiom.
  - $(\forall X)\{\{Na \bullet Xa \bullet (\forall x)[(Nx \bullet Xx) \supset Xf(x)]\} \supset (\forall x)(Nx \supset Xx)\}$

# More Translations

1. Everything has some relation to itself.
   - $(\forall x)(\exists V)Vxx$
2. All people have some property in common.
   - $(\forall x)(\forall y)[(Px \bullet Py) \supset (\exists Y)(Yx \bullet Yy)]$
3. No two people have every property in common.
   - $(\forall x)(\forall y)[(Px \bullet Py) \supset (\exists Z)(Zx \bullet {\sim}Zy)$

# Characterizing Relations

- We can regiment basic characteristics of relations without second-order logic.
- Here are three characteristics of relations, in first-order logic:
  - Reflexivity: $(\forall x)Rxx$
  - Symmetry: $(\forall x)(\forall y)(Rxy \equiv Ryx)$
  - Transitivity: $(\forall x)(\forall y)(\forall z)[(Rxy \bullet Ryz) \supset Rxz]$
- Second-order logic allows us to do more.
- Some relations are transitive.
  - $(\exists X)(\forall x)(\forall y)(\forall z)[(Xxy \bullet Xyz) \supset Xxz]$
- Some relations are symmetric, while some are asymmetric.
  - $(\exists X)(\forall x)(\forall y)(Xxy \equiv Xyx) \bullet (\exists X)(\forall x)(\forall y)(Xxy \equiv {\sim}Xyx)$

# Replacing the Identity Predicate

$$x=y \text{ iff } (\forall X)(Xx \equiv Xy)$$

- Symmetry and reflexivity follow from the symmetry and reflexivity of the biconditional.
- IDi follows from BMP.

# Identity for Properties

- We would like to say something about property identity.
  - For example: There are at least two distinct properties.
  - $(\exists X)(\exists Y)X \neq Y$
- But identity is a relation between singular terms, not predicates.
- And there are no objects attached to the predicates above.
- We can add a quantifier to take care of the singular terms:
  - $(\exists X)(\exists Y)(\exists x){\sim}(Xx \equiv Yx)$
- This only indicates that there are distinct monadic properties.
- What about dyadic properties?
  - $(\exists X)(\exists Y)(\exists x)(\exists y){\sim}(Xxy \equiv Yxy)$
- In order to generalize such claims, higher-order logics are required.

# Higher-Order Logics

- All logics after first-order logic are called higher-order logic
- To create third-order logic, we introduce attributes of attributes.
- All useful properties are desirable.
  - (∀X)(***U***X ⊃ ***D***X)
- A man who possesses all virtues is a virtuous man, but there are virtuous men who do not possess all virtues:
  - (∀x){[Mx • (∀X)(***V***X ⊃ Xx)] ⊃ ***V***x} • (∃x)[Mx • Vx • (∃X)(***V***X • ~Xx)]
- Note the missing objects in the predicate variables above.
- Also, in the latter, the third-order variables are applied both to predicates and terms, which is a category error.
- Cleaning it up would make it messier.
- Yes, I said that.

# Philosophy and Higher- Order Logic

# Against Second-order Logic

- Many philosophers have argued that higher-order logics are not really logic.
- Quine:
  - First-order logic with identity is canonical.
  - Second-order logic is, “Set theory in sheep’s clothing” (*Philosophy of Logic*, p 66).

# Interpretations and Existence

- When we interpret first-order logic, we specify a domain for the variables to range over.
- To be is to be the value of a variable.
- For our most general reasoning, we take an unrestricted domain: the universe, everything there is.
- There are blue hats.
  - $(\exists x)(Bx \bullet Hx)$
- Some properties are shared by two people.
  - $(\exists X)(\exists x)(\exists y)(Px \bullet Py \bullet x{\neq}y \bullet Xx \bullet Xy)$
  - There must exist two people, and there must exist a property.
  - In other words, we need a domain for interpreting the second-order quantifier.
  - That domain will have properties in it.

# The Reification of Properties

- By quantifying over properties, we take properties as objects.
- What are properties?
  - Platonic forms?
  - Eternal ideas?
- Commitments to properties, in addition to objects which have those properties, is metaphysically contentious.
- There are blue things.
- Is blueness also a thing?

# Deflating Second-Order Logic

- We can take properties to be *sets* of objects which have those properties.
- On this extensional interpretation of predicate variables, 'blueness' refers to the collection of all blue things.
- Thus, second-order logic commits us *at least* to the existence of sets.
- We might want to include sets in our ontology.
  - We might think there are mathematical objects.
- We need not include them under the guise of second-order logic.
- We can take them to be values of first-order variables.
- We can count them as among the objects in the universe, in the domain of quantification, rather than sneaking them in through the interpretations of second-order variables.
- We have to look more closely at general principles of theory choice.

# In Favor of Second-Order Logic

- Expressional strength
- Deriving the properties of identity from the second-order axioms, rather than introducing a special predicate with special inferential properties
- Quine favors using schematic predicate letters in lieu of predicate variables.
  - The law of the excluded middle: $P \vee {\sim}P$
- I find this approach disingenuous.
- Schematic letters are really meta-linguistic variables.
- Quine is admitting is that we can not formulate second-order claims in our canonical language.
- We must, instead, ascend to a meta-language, using meta-linguistic variables.

# Derivations in Higher-Order Logics

We will not consider derivations in higher-order logics.

# Review Sessions

Friday in class

Tuesday 12pm - OK?

Final Exam

Wednesday, December 18

2pm - 5pm