

# **Philosophy 240: Symbolic Logic**

Russell Marcus  
Hamilton College  
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Class #29 - Semantics for Predicate Logic

# Theories

- A *theory* is a set of sentences.
- A formal theory is a set of sentences of a formal language.
- We identify a theory by its theorems, the set of sentences provable within that theory.
- Many interesting formal theories are infinite.
  - Rules of inference generate an infinite number of theorems.

# Constructing Formal Theories

1. Specify a language
  - vocabulary
  - formation rules for wffs

# Syntax for **PL** and **M**

## Vocabulary for **PL**

Capital letters A...Z

Five connectives:  $\sim$ ,  $\bullet$ ,  $\vee$ ,  $\supset$ ,  $\equiv$

Punctuation:  $()$ ,  $[\ ]$ ,  $\{ \}$

## Formation rules for Wffs of **PL**

1. A single capital English letter is a wff.
2. If  $\alpha$  is a wff, so is  $\sim\alpha$ .
3. If  $\alpha$  and  $\beta$  are wffs, then so are:  
 $(\alpha \bullet \beta)$   
 $(\alpha \vee \beta)$   
 $(\alpha \supset \beta)$   
 $(\alpha \equiv \beta)$
4. These are the only ways to make wffs.

## Vocabulary for **M**

Capital letters A...Z used as one-place predicates

Lower case letters used as singular terms

a, b, c,...u are constants.

v, w, x, y, z are variables.

Five connectives:  $\sim$ ,  $\bullet$ ,  $\vee$ ,  $\supset$ ,  $\equiv$

Quantifier symbols:  $\exists$ ,  $\forall$

Punctuation:  $()$ ,  $[\ ]$ ,  $\{ \}$

## Formation Rules for Wffs of **M**

1. A predicate (capital letter) followed by a constant or variable (lower-case letter) is a wff.
2. For any variable  $\beta$ , if  $\alpha$  is a wff that does not contain either ' $(\exists\beta)$ ' or ' $(\forall\beta)$ ', then ' $(\exists\beta)\alpha$ ' and ' $(\forall\beta)\alpha$ ' are wffs.
3. If  $\alpha$  is a wff, so is  $\sim\alpha$ .
4. If  $\alpha$  and  $\beta$  are wffs, then so are:  
 $(\alpha \bullet \beta)$   
 $(\alpha \vee \beta)$   
 $(\alpha \supset \beta)$   
 $(\alpha \equiv \beta)$
5. These are the only ways to make wffs.

# Constructing Formal Theories

## 1. Specify a language

- vocabulary
- formation rules for wffs

## 2. Add formation rules for wffs.

To construct a formal theory, we select some of the wffs as our theorems.

- Different theories can be written in the same language.

## 3a. Specify theorems

- We can list all of our theorems: finite theories
- We can specify axioms and rules of inference.
  - Proof Theory

## 3b. Provide a semantics for the theory.

- Specify truth conditions and truth values for wffs
- Model Theory (Truth)

# Goodness for Theories

- In some theories, the provable theorems are exactly the same as the true wffs.
  - Proof theory and semantics align!
  - *Soundness*: all the provable theorems are true
  - *Completeness*: all the truths are provable
- In more sophisticated theories, proof separates from truth
  - Gödel's first incompleteness theorem
  - For most interesting theories beyond PL and M, there are true sentences of the theory that are not provable within the system.
    - This is a mind-blowingly awesome result.
  - Model theory and proof theory come apart.



# Semantics and Proof Theory for PL

- In **PL**, our semantics used truth tables.
  - Interpretations of **PL**
    - **Step 1. Assign 1 or 0 to each atomic sentence.**
      - Only finitely many ( $2^{26} = \sim 6.7$  million) possible interpretations in our language.
      - We could use a language with infinitely many simple terms:  $P, P', P'', P''', P'''' \dots$
    - **Step 2. Assign truth values to complex propositions by combining, according to the truth table definitions, the truth values of the atomic sentences.**
  - In **M**, and the other languages of predicate logic, the semantics are more complicated.
    - interpretation, satisfaction, logical truth, validity
- In proof theory, we construct a system of inference using the formal language we have specified.
  - In **PL**, our proof system was our eighteen rules of natural deduction.
    - no axioms
    - conditional and indirect proof
  - Other proof systems use axioms.

# Semantics for M

- Separating the syntax of our language from its semantics allows us to treat our formal languages as completely uninterpreted.
  - Intuitively, we know what the logical operators mean.
  - But until we specify a formal interpretation, we are free to interpret them as we wish.
- Our constants and predicates and quantifiers are, as far as the syntax of our language specifies, uninterpreted.
- To look at the logical properties of the language, we construct formal semantics.
- The first step in formal semantics is to show how to provide an interpretation of the language.
- Then, we can determine the logical truths.
  - The wffs that come out as true under every interpretation.



# Interpretations of M

- To define an interpretation in M, or in any of its extensions, we have to specify how to handle constants, predicates, and quantifiers.
  - We use some set theory in our meta-language.
- Step 1. Specify a set to serve as a domain of interpretation (or quantification).
  - We can consider small finite domains  
Domain<sub>1</sub> = {1, 2, 3}  
Domain<sub>2</sub> = {Barack Obama, Hillary Clinton, Joe Biden}.
  - We can consider larger domains, like a universe of everything.
- Step 2. Assign a member of the domain to each constant.
  - a: 1; b: 2; c: 3
  - a: Obama; b: Clinton
- Step 3. Assign some set of objects in the domain to each predicate.
  - 'Ex' may stand for 'x has been elected president'
  - In Domain<sub>1</sub>, the interpretation of 'Ex' will be empty.
  - In Domain<sub>2</sub>, it will be {Barack Obama}.
- **Step 4.** Use the customary truth tables for the interpretation of the connectives.

# Satisfaction and Truth-for-an-Interpretation

- Objects in the domain may satisfy predicates.
  - Ordered n-tuples may satisfy relations.
- A wff will be satisfiable if there are objects in the domain of quantification which satisfy the predicates indicated in the wff.
  - A universally quantified sentence is satisfied if it is satisfied by all objects in the domain.
  - An existentially quantified sentence is satisfied if it is satisfied by some object in the domain.
- A wff will be true-for-an-interpretation if all objects in the domain of quantification satisfy the predicates indicated in the wff.
- We call an interpretation on which all of a set of statements come out true a *model*.

# An Interpretation of a Theory

- Theory
  1.  $Pa \bullet Pb$
  2.  $Wa \bullet \sim Wb$
  3.  $(\exists x)Px$
  4.  $(\forall x)Px$
  5.  $(\forall x)(Wx \supset Px)$
  6.  $(\forall x)(Px \supset Wx)$
- Step 1: Specify a set to serve as a domain of interpretation, or domain of quantification.
  - Domain: {Bob Simon, Rick Werner, Katheryn Doran, Todd Franklin, Marianne Janack, Russell Marcus, Theresa Lopez}
- Step 2: Assign a member of the domain to each constant.
  - a: Katheryn Doran
  - b: Bob Simon

Notice: no other constants in our theory  
Some objects remain without names
- Step 3: Assign some set of objects in the domain to each predicate.
  - $Px$ : {Bob Simon, Rick Werner, Katheryn Doran, Todd Franklin, Marianne Janack, Russell Marcus, Theresa Lopez}
  - $Wx$ : {Katheryn Doran, Marianne Janack, Theresa Lopez}

# Models in M

- We call an interpretation on which all of a set of given statements come out true a *model*.
- Given our interpretations of the predicates, not every sentence in our set is satisfied.
  - 1-5 are satisfied.
  - 6 is not.
- If we were to delete sentence 6 from our list, our interpretation would be a model.

1.  $Pa \bullet Pb$
2.  $Wa \bullet \sim Wb$
3.  $(\exists x)Px$
4.  $(\forall x)Px$
5.  $(\forall x)(Wx \supset Px)$
6.  $(\forall x)(Px \supset Wx)$

Domain: {Bob Simon, Rick Werner, Katheryn Doran, Todd Franklin, Marianne Janack, Russell Marcus, Theresa Lopez}  
a: Katheryn Doran  
b: Bob Simon  
Px: {Bob Simon, Rick Werner, Katheryn Doran, Todd Franklin, Marianne Janack, Russell Marcus, Theresa Lopez}  
Wx: {Katheryn Doran, Marianne Janack, Theresa Lopez}

# Constructing a Model

- Theory

1.  $(\forall x)(Px \supset Qx)$
2.  $(\exists x)(Px \bullet Rx)$
3.  $(\exists x)(Qx \bullet \sim Px)$
4.  $(\exists x)(Qx \bullet \sim Rx)$
5.  $(Pa \bullet Pb) \bullet Qc$

- **Step 1.** Specify a set to serve as a domain of interpretation, or domain of quantification.

Domain = {Persons}

- **Step 2.** Assign a member of the domain to each constant.

a = Barack Obama

b = Condoleezza Rice

c = Neytiri (from *Avatar*)



- **Step 3.** Assign some set of objects in the domain to each predicate.

$Px = \{\text{Human Beings}\}$

$Qx = \{\text{Persons}\}$

$Rx = \{\text{Males}\}$

- **Step 4.** Use the customary truth tables for the interpretation of the connectives.

# Logical Truth in M

- A wff of M will be logically true if it is true for every interpretation.
- For **PL**, the notion of logical truth was simple.
  - Just look at the truth tables.
- For **M**, and even more so for **F** (full first-order logic), the notion of logical truth is naturally complicated by the fact that we are analyzing parts of propositions.
- Here are two logical truths of **M**:
  - $(\forall x)(Px \vee \sim Px)$
  - $Pa \vee [(\forall x)Px \supset Qa]$
- As in **PL**, we can show that a wff is a theorem (logical truth) proof-theoretically and model-theoretically.

# Proof-Theoretic Argument

$$(\forall x)(Px \vee \sim Px)$$

- |                                       |             |
|---------------------------------------|-------------|
| 1. $\sim(\forall x)(Px \vee \sim Px)$ | AIP         |
| 2. $(\exists x)\sim(Px \vee \sim Px)$ | 1, QE       |
| 3. $\sim(Pa \vee \sim Pa)$            | 2, EI       |
| 4. $\sim Pa \bullet \sim\sim Pa$      | 3, DM       |
| 5. $(\forall x)(Px \vee \sim Px)$     | 1-4, IP, DN |

# Model-Theoretic Argument

$$Pa \vee [(\forall x)Px \supset Qa]$$

- Consider an interpretation on which ' $Pa \vee [(\forall x)Px \supset Qa]$ ' is false.
- The object assigned to 'a' will not be in the set assigned to 'Px', and there is some counterexample to ' $(\forall x)Px \supset Qa$ '.
- But, any counter-example to a conditional statement has to have a true antecedent.
- So, every object in the domain will have to be in the set assigned to 'Px'.
  - Tilt
- So, no interpretation will make that sentence false.
- So, ' $Pa \vee [(\forall x)Px \supset Qa]$ ' is logically true.



# Another Logical Truth

$$(\exists x)Px \vee (\forall x)(Qx \supset \sim Px)$$

- Try it both ways!

# Validity

- A valid argument will have to be valid under any interpretation, using any domain.
- Our proof system has given us ways to show that an argument is valid.
- But when we introduced our system of inference for **PL**, we already had a way of distinguishing the valid from the invalid arguments, using truth tables.
- In **M**, we need a corresponding method for showing that an argument is invalid.
- An invalid argument will have counter-examples, interpretations on which the premises come out true and the conclusion comes out false.

# Coming Up

- Monday
  - Invalid Arguments in Predicate Logic
  - Constructing Counterexamples
- But first: Friday
  - Philosophy Friday
  - Existential Quantifiers and Ontological Commitments