

Philosophy 240
Symbolic Logic

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Class #20: On Axiomatic Systems

Hunter's System PS

With axiom schemata

- Three axiom schemata:

PS1: $\alpha \supset (\beta \supset \alpha)$

PS2: $(\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$

PS3: $(\sim\alpha \supset \sim\beta) \supset (\beta \supset \alpha)$

Any formula of the language PS of any of the forms PS1, PS2, or PS3 is an axiom of PS.

– Infinitely many axioms of PS

- One rule of inference, modus ponens.

If α and β are formulas, β is a consequence in system PS of α and $\alpha \supset \beta$.

$\alpha, (\alpha \supset \beta) \vdash_{\text{PS}} \beta$

Hunter's System PS

With a substitution rule

- Three axioms

PS1*: $P \supset (Q \supset P)$

PS2*: $(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$

PS3*: $(\sim P \supset \sim Q) \supset (Q \supset P)$

- Two rules of inference

MP: $\alpha, (\alpha \supset \beta) \vdash_{PS} \beta$

Substitution: For any wff α , and any wff β appearing as a subformula in wff δ : If $\vdash_{PS} \delta$, then $\vdash_{PS} \delta_{\alpha}^{\beta}$

– where ' δ_{α}^{β} ' is the result of substituting ' α ' for ' β ' throughout.

Hunter's PS and Our PL

- Hunter's axiomatic system is provably equivalent to our more-familiar natural deduction system PL.
 - A metalogical result (the deduction theorem) allows us to move between the two kinds of systems.
- Frege's original *Begriffsschrift* used a Hilbert-style axiomatic system, with a different axiomatization.
 - See the appendices in Richard Mendelsohn's *The Philosophy of Gottlob Frege*, for a translation of the *Begriffsschrift* into modern notation.
- Natural deduction systems like PS in *What Follows* are due largely to Gerhard Gentzen's work in the 1930s and 1940s.
 - There is a paper on the history of natural deduction by Pelletier which might be worth a look for a term paper.
- Completeness:
 - Frege's axiomatization, Hunter's PS, and standard systems of natural deduction are all complete.
 - All logical truths are provable within the system.
- One's choice of system is thus arbitrary among the various complete systems.

A Derivation in PS

PS1: $\alpha \supset (\beta \supset \alpha)$

PS2: $(\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$

PS3: $(\sim\alpha \supset \sim\beta) \supset (\beta \supset \alpha)$

■ A1: $\vdash_{PS} P \supset P$

1. $P \supset ((P \supset P) \supset P)$

PS1

2. $(P \supset ((P \supset P) \supset P)) \supset ((P \supset (P \supset P)) \supset (P \supset P))$

PS2

3. $(P \supset (P \supset P)) \supset (P \supset P)$

MP, 1, 2

4. $P \supset (P \supset P)$

PS1

5. $P \supset P$

3, 4, MP

■ QED

A Further Derivation in PS

PS1: $\alpha \supset (\beta \supset \alpha)$

PS2: $(\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$

PS3: $(\sim\alpha \supset \sim\beta) \supset (\beta \supset \alpha)$

A1: $P \supset P$

■ A2: $\vdash_{PS} \sim P \supset (P \supset P)$

1. $P \supset P$

2. $(P \supset P) \supset (\sim P \supset (P \supset P))$

3. $\sim P \supset (P \supset P)$

Ex A1

PS 1

MP, 1, 2

■ QED

A Longer Proof in PS

■ $\vdash_{PS} (P \supset (Q \supset R)) \supset (Q \supset (P \supset R))$

1. $Q \supset (P \supset Q)$
2. $(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$
3. $[(P \supset Q) \supset (P \supset R)] \supset [Q \supset ((P \supset Q) \supset (P \supset R))]$
4. $\{[(P \supset Q) \supset (P \supset Q')] \supset (Q \supset ((P \supset Q) \supset (P \supset R)))\} \supset \{(P \supset (Q \supset R)) \supset [((P \supset Q) \supset (P \supset R)) \supset (Q \supset ((P \supset Q) \supset (P \supset R)))]\}$
5. $(P \supset (Q \supset R)) \supset [((P \supset Q) \supset (P \supset R)) \supset (Q \supset ((P \supset Q) \supset (P \supset R)))]$
6. $\{(P \supset (Q \supset R)) \supset [((P \supset Q) \supset (P \supset R)) \supset (Q \supset ((P \supset Q) \supset (P \supset R)))]\} \supset \{(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))\} \supset [(P \supset (Q \supset R)) \supset (Q \supset ((P \supset Q) \supset (P \supset R)))]$
7. $[(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))] \supset [(P \supset (Q \supset R)) \supset (Q \supset ((P \supset Q) \supset (P \supset R)))]$
8. $(P \supset (Q \supset R)) \supset [Q \supset ((P \supset Q) \supset (P \supset R))]$
9. $[Q \supset ((P \supset Q) \supset (P \supset R))] \supset [(Q \supset (P \supset Q)) \supset (Q \supset (P \supset R))]$
10. $[(Q \supset ((P \supset Q) \supset (P \supset R))) \supset ((Q \supset (P \supset Q)) \supset (Q \supset (P \supset R)))] \supset \{(P \supset (Q \supset R)) \supset [(Q \supset ((P \supset Q) \supset (P \supset R))) \supset ((Q \supset (P \supset Q)) \supset (Q \supset (P \supset R)))]\}$
11. $(P \supset (Q \supset R)) \supset [(Q \supset ((P \supset Q) \supset (P \supset R))) \supset ((Q \supset (P \supset Q)) \supset (Q \supset (P \supset R)))]$
12. $\{(P \supset (Q \supset R)) \supset [(Q \supset ((P \supset Q) \supset (P \supset R))) \supset ((Q \supset (P \supset Q)) \supset (Q \supset (P \supset R)))]\} \supset \{(P \supset (Q \supset R)) \supset (Q \supset ((P \supset Q) \supset (P \supset R)))\} \supset [(P \supset (Q \supset R)) \supset ((Q \supset (P \supset Q)) \supset (Q \supset (P \supset R)))]$
13. $[(P \supset (Q \supset R)) \supset (Q \supset ((P \supset Q) \supset (P \supset R)))] \supset [(P \supset (Q \supset R)) \supset ((Q \supset (P \supset Q)) \supset (Q \supset (P \supset R)))]$
14. $(P \supset (Q \supset R)) \supset [(Q \supset (P \supset Q)) \supset (Q \supset (P \supset R))]$
15. $\{(P \supset (Q \supset R)) \supset [(Q \supset (P \supset Q)) \supset (Q \supset (P \supset R))]\} \supset \{[(P \supset (Q \supset R)) \supset (Q \supset (P \supset Q))] \supset [(P \supset (Q \supset R)) \supset (Q \supset (P \supset R))]\}$
16. $[(P \supset (Q \supset R)) \supset (Q \supset (P \supset Q))] \supset [(P \supset (Q \supset R)) \supset (Q \supset (P \supset R))]$
17. $(Q \supset (P \supset Q)) \supset [(P \supset (Q \supset R)) \supset (Q \supset (P \supset Q))]$
18. $(P \supset (Q \supset R)) \supset (Q \supset (P \supset Q))$
19. $(P \supset (Q \supset R)) \supset (Q \supset (P \supset R))$

■ QED