Philsoophy 240: Symbolic Logic Hamilton College
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A note on the translation of 'and' in antecedents of universally quantified formulae of **M**
(aka my existential crisis)

Our discussion in class was triggered by the translation into **M** of 3.1c: 10: Not all dogs and cats like humans. I claimed the equivalence of two formulae which Noah pointed out had different existential implications. I was wrong about their equivalence and I'll here provide a more careful account of the relation between the given pairs and some related ones. Some of this explanation requires some machinery we don't have yet: rules governing quantifiers, changes of quantifiers, and CP and IP with quantifiers. I promise that everything I'm doing is kosher and that it should make more sense after we get the derivation rules under our belts. Don't fret too much about the details of the derivations at this point.

Let's start with a an example nearby to 3.1c: 10, but more simple: 3.1c: 8: All cats and dogs have whiskers. We can regiment 3.1c: 8: in **M** in two equivalent ways:

A1 (∀x)[(Cx ∨ Dx) ⊃ Wx]
A2 (∀x)(Cx ⊃ Wx) • (∀x)(Dx ⊃ Wx)

To show that A1 and A2 are equivalent, we can do two derivations: A1 from A2 and A2 from A1. I'll start deriving A2 from A1:

1. (∀x)[(Cx ∨ Dx) ⊃ Wx]	Premise
| 2. ~[(∀x)(Cx ⊃ Wx) • (∀x)(Dx ⊃ Wx)]	AIP
| 3. ~(∀x)(Cx ⊃ Wx) ∨ ~ (∀x)(Dx ⊃ Wx)	2, DM
| 4. (∃x)~(Cx ⊃ Wx) ∨ (∃x)~(Dx ⊃ Wx)	3, QE
| 5. (∃x)~(~Cx ∨ Wx) ∨ (∃x)~(~Dx ∨ Wx)	4, Impl
| 6. (∃x)(Cx • ~Wx) ∨ (∃x)(Dx • ~Wx)	5, DM, DN
| | 7. (∃x)(Cx • ~Wx)	AIP
| | 8. Ca • ~Wa	7, EI
| | 9. (Ca ∨ Da) ⊃ Wa	1, UI
| | 10. Ca	8, Simp
| | 11. Ca ∨ Da	10, Add
| | 12. Wa	9, 11, MP
| | 13. ~Wa	8, Com, Simp
| | 14. Wa • ~Wa	12, 13, Conj
| 15. ~(∃x)(Cx • ~Wx)	7-14, IP
| 16. (∃x)(Dx • ~Wx)	6, 15, DS
| 17. Db • ~Wb	16, EI
| 18. Db	17, Simp
| 19. (Cb ∨ Db) ⊃ Wb	1, UI
| 20. Cb ∨ Db	18., Add Com
| 21. Wb	19, 20, MP
| 22. ~Wb	17, Com, Simp
| 23. Wb • ~Wb	21, 22, Conj
24. [(∀x)(Cx ⊃ Wx) • (∀x)(Dx ⊃ Wx)]	2-23, IP, DN

QED

Now, let's derive A1 from A2.

1. $(\forall x)(Cx \supset Wx) \bullet (\forall x)(Dx \supset Wx)$	Premise
2. $(\forall x)(Cx \supset Wx)$	1, Simp
3. $Cx \supset Wx$	2, UI
4. $(\forall x)(Dx \supset Wx)$	1, Com, Simp
5. $Dx \supset Wx$	4, UI
6. $(Cx \supset Wx) \bullet (Dx \supset Wx)$	3, 5, Conj
7. $(\sim Cx \vee Wx) \bullet (\sim Dx \vee Wx)$	6, Impl
8. $(Wx \vee \sim Cx) \bullet (Wx \vee \sim Dx)$	7, Com
9. $Wx \vee (\sim Cx \bullet \sim Dx)$	8, Dist
10. $Wx \vee \sim(Cx \vee Dx)$	9, DM
11. $\sim(Cx \vee Dx) \vee Wx$	10, Com
12. $(Cx \vee Dx) \supset Wx$	11, Impl
13. $(\forall x)[(Cx \vee Dx) \supset Wx]$	12, UG

QED

Since A1 and A2 are derivable from each other, they are logically equivalent. (Either that or our logic is inconsistent, which it's not.) Notice that this pair doesn't evoke Noah's concern. In both cases, we have universally quantified formula. All things which are cats and all things which are dogs must have whiskers for both statements to be true.

Now, let's look at a related pairs of claims which invoke existential, rather than universal, quantifiers.

E1 $(\exists x)[(Cx \vee Dx) \bullet Bx]$
E2 $(\exists x)(Cx \bullet Bx) \bullet (\exists x)(Dx \bullet Bx)$

Let's take 'Cx' to stand for 'x is a cat', 'Dx' to stand for 'x is a dog', and 'Bx' to stand for 'x is brown'. Then E2 says that some cats are brown and some dogs are brown. As Noah pointed out, this claim seems to require at least two objects: a brown cat and a brown dog. E1, in contrast, seems to require only one object, either a brown cat or a brown dog. So E1 and E2 don't appear to be logically equivalent. Fortunately for our logic, they are not. E1 follows easily from E2.

1. $(\exists x)(Cx \bullet Bx) \bullet (\exists x)(Dx \bullet Bx)$	Premise
2. $(\exists x)(Cx \bullet Bx)$	1, Simp
3. $Ca \bullet Ba$	2, EI
4. Ca	3, Simp
5. $Ca \vee Da$	4, Add
6. Ba	3, Com, Simp
7. $(Ca \vee Da) \bullet Ba$	5, Conj
8. $(\exists x)[(Cx \vee Dx) \bullet Bx]$	7, EG

QED

But E1 does not entail E2. Since we haven't talked about how to show arguments invalid in predicate logic and it's rather different from the propositional logic case, I won't show it here in detail. The short version is this: in a world in which there is only a brown cat, E1 is true but E2 is false. The inference from E1 to E2 has a true premise but a false conclusion; The argument is invalid and so E1 and E2 are not equivalent.

Now, let's get to our original problem: 3.1c: 10: Not all dogs and cats like humans. The two versions under consideration are:

V1 $\sim(\forall x)[(Dx \vee Cx) \supset Lx]$
V2 $\sim(\forall x)(Dx \supset Lx) \bullet \sim(\forall x)(Cx \supset Lx)$

Given the relationship between A1 and A2, one might (if one were, say, a bald, middle-aged logic teacher at a well-respected small liberal arts college in Central New York on a chilly October morning) glibly assert the equivalence of V1 and V2. But one would in that case be wrong. The problem is that V1 and V2, while looking a bit like the universally quantified and logically equivalent formulae A1 and A2, are really existential formulae, equivalent, respectively, to the pair V1E and V2E.

V1E $(\exists x)[(Dx \vee Cx) \bullet \sim Lx]$
V2E $(\exists x)(Dx \bullet \sim Lx) \bullet (\exists x)(Cx \bullet \sim Lx)$

As in the relevantly similar cases of E1 and E2, V1E and V2E are not equivalent. We can derive V1E from V2E just as we derived E1 from E2. But V2E does not follow from V1E since, as Noah pointed out, it requires two thing where V1E does not.

Actually, that's a little too quick. V2E could be true if there were just one thing: a dog-cat who does not like humans. (Let's set aside the question how this dog-cat could dislike humans if there aren't any!) In other words, 'D' and 'C' could be different names for the same property. There's nothing in the logic which prevents that. The problem which decisively shows that V1E and V2E are not equivalent is that we can provide an interpretation on which V1E is true and V2E is false. That's all we need to show that the inference from V1E to V2E is invalid and thus that the two are not logically equivalent.

Finally, let's take up Andrew's sage suggestion that we can fix the problem by changing V2E to a disjunction. In class, this suggestion struck me as unmotivated and *ad hoc*. So much for *my* logical intuitions. To be clear, Andrew's suggestion was that V1ED and V2ED are logically equivalent.[1]

V1ED $(\exists x)[(Dx \vee Cx) \bullet \sim Lx]$
V2ED $(\exists x)(Dx \bullet \sim Lx) \vee (\exists x)(Cx \bullet \sim Lx)$

Well, what do you think? Do you think I'm going to spend my valuable time trying to type up derivations of each from the other? I could be spending time with my children, you know. Actually, they're at Hebrew school and I don't have to pick them up for another twenty minutes. I could go lie down and rest for a moment, but...see, I wanted to start the derivations on the next page and it seemed...

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[1] Technically, I think, Andrew's suggestion was that V1D and V2D are equivalent.

V1D $\sim(\forall x)[(Dx \vee Cx) \supset Lx]$
V2D $\sim(\forall x)(Dx \supset Lx) \vee \sim(\forall x)(Cx \supset Lx)$

But, it's going to be easier to work with the existential equivalents of each and the same point holds. I'll be happy to return to the equivalents of V1 with V1E, V2 with V2E, V1D with V1ED, and V2D with V2Ed later, after we get to quantifier exchange. At that point, though, the equivalence will be so obvious that I'll probably be wasting our time. Plus, something about beating a dead horse, which is really not a pleasant metaphor at all.

V1ED $\vdash$ V2ED

1. $(\exists x)[(Dx \vee Cx) \bullet {\sim}Lx]$	Premise
2. ${\sim}[(\exists x)(Dx \bullet {\sim}Lx) \vee (\exists x)(Cx \bullet {\sim}Lx)]$	AIP
3. ${\sim}(\exists x)(Dx \bullet {\sim}Lx) \bullet {\sim}(\exists x)(Cx \bullet {\sim}Lx)$	2, DM
4. $(\forall x){\sim}(Dx \bullet {\sim}Lx) \bullet (\forall x){\sim}(Cx \bullet {\sim}Lx)$	3, QE (OK, I’m showing it anyway)
5. $(\forall x)({\sim}Dx \vee Lx) \bullet (\forall x)({\sim}Cx \vee Lx)$	4, DM, DN
6. $(\forall x)(Dx \supset Lx) \bullet (\forall x)(Cx \supset Lx)$	5, Impl
7. $(Da \vee Ca) \bullet {\sim}La$	1, EI
8. $(\forall x)(Dx \supset Lx)$	6, Simp
9. $Da \supset La$	8, UI
10. $(\forall x)(Cx \supset Lx)$	6, Com, Simp
11. $Ca \supset La$	10, UI
12. $(Da \supset La) \bullet (Ca \supset La)$	9, 11, Conj
13. $Da \vee Ca$	7, Simp
14. $La \vee La$	12, 13, CD
15. La	14, Taut
16. ${\sim}La$	7, Com, Simp
17. $La \bullet {\sim}La$	15, 16, Conj
18. $(\exists x)(Dx \bullet {\sim}Lx) \vee (\exists x)(Cx \bullet {\sim}Lx)$	2-17, IP, DN

QED

V2ED $\vdash$ V1ED

1. $(\exists x)(Dx \bullet {\sim}Lx) \vee (\exists x)(Cx \bullet {\sim}Lx)$	Premise
2. ${\sim}(\exists x)[(Dx \vee Cx) \bullet {\sim}Lx]$	AIP
3. $(\forall x){\sim}[(Dx \vee Cx) \bullet {\sim}Lx]$	2, QE
4. $(\forall x)[{\sim}(Dx \vee Cx) \vee Lx]$	3, DM, DN
5. $(\forall x)[(Dx \vee Cx) \supset Lx]$	4, Impl
6. $(\exists x)(Dx \bullet {\sim}Lx)$	AIP
7. $Da \bullet {\sim}La$	6, EI
8. $(Da \vee Ca) \supset La$	5, UI
9. Da	7, Simp
10. $Da \vee Ca$	9, Add
11. La	8, 10, MP
12. ${\sim}La$	7, Com, Simp
13. $La \bullet {\sim}La$	11, 12, Conj
14. ${\sim}(\exists x)(Dx \bullet {\sim}Lx)$	6-13, IP
15. $(\exists x)(Cx \bullet {\sim}Lx)$	1, 14, DS
16. $Cb \bullet {\sim}Lb$	15, EI
17. $(Db \vee Cb) \supset Lb$	5, EI
18. Cb	16, Simp
19. $Db \vee Cb$	18, Add, Com
20. Lb	17, 19, MP
21. ${\sim}Lb$	16, Com, Simp
22. $Lb \bullet {\sim}Lb$	20, 21, Conj
23. $(\exists x)[(Dx \vee Cx) \bullet {\sim}Lx]$	2-22, IP, DN

QED

So,V1ED and V2ED are equivalent.