Philosophy 240
Symbolic Logic

Russell Marcus
Hamilton College
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Class 34
More Translation Using Relational Predicates
Rules of Passage

Quantifiers: Narrow and Wide Scope

- Wide scope: standing in front of the proposition
  - $(\forall x)(\forall y)[Px \supset (Qy \supset Rxy)]$
  - $(\exists x)(\exists y)[(Px \bullet Py) \bullet Lxy]$
- Narrow scope: located inside the proposition
  - $(\forall x)[Px \supset (\forall y)(Qy \supset Rxy)]$
  - $(\exists x)[Px \bullet (\exists y)(Py \bullet Lxy)]$
- Mostly, it is good form to keep narrow scope.

Moving Universal Quantifiers

- In some cases, we can move quantifiers around without much worry.
  - If all quantifiers are universal, we can pull them in or out.
  - Be careful not to accidentally bind any variables!
- Everyone loves everyone
  - $(\forall x)[Px \supset (\forall y)(Py \supset Lxy)]$
  - $(\forall x)(\forall y)[(Px \cdot Py) \supset Lxy]$
  - $(\forall y)(\forall x)[(Px \cdot Py) \supset Lxy]$

Moving Existential Quantifiers

Someone loves someone

- (∃x)[Px · (∃y)(Py · Lxy)]
- (∃x)(∃y)[(Px · Py) · Lxy]
- (∃y)(∃x)[(Px · Py) · Lxy]

Mixing Quantifiers

- None of the following examples are equivalent:
  - Everyone loves someone: $(\forall x)(\exists y)[Px \supset (Py \cdot Lxy)]$
  - Everyone is loved by someone: $(\forall x)(\exists y)[Px \supset (Py \cdot Lyx)]$
  - Someone loves everyone: $(\exists x)(\forall y)[Px \cdot (Py \supset Lxy)]$
  - Someone is loved by everyone: $(\exists x)(\forall y)[Px \cdot (Py \supset Lyx)]$
- The first word in each translation above corresponds to the leading quantifier.
- The connectives which directly follow the 'Px' and the 'Py' are determined by the quantifier binding that variable.

Using Narrow Scope

- Everyone loves someone.
  $(\forall x)[Px \supset (\exists y)(Py \cdot Lxy)]$
- Everyone is loved by someone.
  $(\forall x)[Px \supset (\exists y)(Py \bullet Lyx)]$
- Someone loves everyone.
  $(\exists x)[Px \cdot (\forall y)(Py \supset Lxy)]$
- Someone is loved by everyone.
  $(\exists x)[Px \cdot (\forall y)(Py \supset Lyx)]$

Moving Mixed Quantifiers: A Problem

- The following sentences are *not* equivalent
  - $(\forall x)[(\exists y)Lxy \supset Hx]$

    For any x, if there is a y that x loves, then x is happy.

    All lovers are happy.
  - $(\forall x)(\exists y)(Lxy \supset Hx)$

    For any x, there is a y such that if x loves y then x is happy.
- The first does not commit to the existence of something that, by being loved, makes a person happy.
- The second does.

Prenex Normal Form (PNF)

- Some metalogical proofs require all statements of **F** to be written with all quantifiers having wide scope.
- A sentence is in Prenex Normal Form (PNF) if all of its quantifiers are in the front, having wide scope.
- Rules of Passage allow us to transform all statements of **F** into PNF.
- They are rules of replacement.
- I will not require that you use them in proofs.
- They may be useful in learning how to translate.

Rules of Passage

- For all variables α and all formulas Γ and Δ:

  RP1: $(\exists\alpha)(\Gamma \vee \Delta) \rightleftarrows (\exists\alpha)\Gamma \vee (\exists\alpha)\Delta$

  RP2: $(\forall\alpha)(\Gamma \bullet \Delta) \rightleftarrows (\forall\alpha)\Gamma \bullet (\forall\alpha)\Delta$

- For all variables α, all formulas Γ containing α, and all formulas Δ not containing α:

  RP3: $(\exists\alpha)(\Delta \bullet \Gamma\alpha) \rightleftarrows \Delta \bullet (\exists\alpha)\Gamma\alpha$

  RP4: $(\forall\alpha)(\Delta \bullet \Gamma\alpha) \rightleftarrows \Delta \bullet (\forall\alpha)\Gamma\alpha$

  RP5: $(\exists\alpha)(\Delta \vee \Gamma\alpha) \rightleftarrows \Delta \vee (\exists\alpha)\Gamma\alpha$

  RP6: $(\forall\alpha)(\Delta \vee \Gamma\alpha) \rightleftarrows \Delta \vee (\forall\alpha)\Gamma\alpha$

  RP7: $(\exists\alpha)(\Delta \supset \Gamma\alpha) \rightleftarrows \Delta \supset (\exists\alpha)\Gamma\alpha$

  RP8: $(\forall\alpha)(\Delta \supset \Gamma\alpha) \rightleftarrows \Delta \supset (\forall\alpha)\Gamma\alpha$

  RP9: $(\exists\alpha)(\Gamma\alpha \supset \Delta) \rightleftarrows (\forall\alpha)\Gamma\alpha \supset \Delta$

  RP10: $(\forall\alpha)(\Gamma\alpha \supset \Delta) \rightleftarrows (\exists\alpha)\Gamma\alpha \supset \Delta$

(Slightly) Friendlier Versions of the Rules of Passage

RP1: $(\exists x)(Px \vee Qx) \rightleftarrows (\exists x)Px \vee (\exists x)Qx$

RP2: $(\forall x)(Px \bullet Qx) \rightleftarrows (\forall x)Px \bullet (\forall x)Qx$

RP3: $(\exists x)(\mathscr{F} \bullet Px) \rightleftarrows \mathscr{F} \bullet (\exists x)Px$

RP4: $(\forall x)(\mathscr{F} \bullet Px) \rightleftarrows \mathscr{F} \bullet (\forall x)Px$

RP5: $(\exists x)(\mathscr{F} \vee Px) \rightleftarrows \mathscr{F} \vee (\exists x)Px$

RP6: $(\forall x)(\mathscr{F} \vee Px) \rightleftarrows \mathscr{F} \vee (\forall x)Px$

RP7: $(\exists x)(\mathscr{F} \supset Px) \rightleftarrows \mathscr{F} \supset (\exists x)Px$

RP8: $(\forall x)(\mathscr{F} \supset Px) \rightleftarrows \mathscr{F} \supset (\forall x)Px$

RP9: $(\exists x)(Px \supset \mathscr{F}) \rightleftarrows (\forall x)Px \supset \mathscr{F}$

RP10: $(\forall x)(Px \supset \mathscr{F}) \rightleftarrows (\exists x)Px \supset \mathscr{F}$

Examples

RP4: $(\forall\alpha)(\Delta \bullet \Gamma\alpha) \rightleftarrows \Delta \bullet (\forall\alpha)\Gamma\alpha$
RP8: $(\forall\alpha)(\Delta \supset \Gamma\alpha) \rightleftarrows \Delta \supset (\forall\alpha)\Gamma\alpha$
RP9: $(\exists\alpha)(\Gamma\alpha \supset \Delta) \rightleftarrows (\forall\alpha)\Gamma\alpha \supset \Delta$
RP10: $(\forall\alpha)(\Gamma\alpha \supset \Delta) \rightleftarrows (\exists\alpha)\Gamma\alpha \supset \Delta$

- Using RP4:
  - $(\exists x)[Px \bullet (\forall y)(Qy \supset Rxy)]$
  - $(\exists x)(\forall y)[Px \bullet (Qy \supset Rxy)]$
- Using RP8:
  - $(\exists x)(\forall y)[Px \supset (Qy \supset Rxy)]$
  - $(\exists x)[Px \supset (\forall y)(Qy \supset Rxy)]$
- Using RP9:
  - $(\forall x)(\exists y)(Lxy \supset Hx)$
  - $(\forall x)[(\forall y)Lxy \supset Hx]$
- Using RP10:
  - $(\forall x)[(\exists y)Lxy \supset Hx]$
  - $(\forall x)(\forall y)(Lxy \supset Hx)$
- Also Using RP10:
  - $(\forall x)[Px \supset (\exists y)Qy]$
  - $(\exists x)Px \supset (\exists y)Qy$

Proving RP10

RP10: $(\forall\alpha)(\Gamma\alpha \supset \Delta) \rightleftarrows (\exists\alpha)\Gamma\alpha \supset \Delta$

- Consider first what happens when Δ is true, and then when Δ is false.
- If Δ is true, then both formulas will turn out to be true.
  - The consequent of the formula on the right is just Δ.
  - So, if Δ is true, the whole formula on the right will be true.
  - On the left, $\Gamma\alpha \supset \Delta$ will be true for every instance of α, since the consequent is true.
  - So, the universal generalization of each such formula will be true.
- If Δ is false, then the truth value of each formula will depend.
  - To show that the truth values of each formula will be the same, we will show that the formula on the right is true in every case that the formula on the left is true and that the formula on the left is true in every case that the formula on the right is true.
  - If the formula on the left turns out to be true when Δ is false, it must be because Γα is false, for every α.
  - But then, $(\exists\alpha)\Gamma\alpha$ will be false, and so the formula on the right turns out to be true.
  - If the formula on the right turns out to be true, then it must be because $(\exists\alpha)\Gamma\alpha$ is false.
  - And so, there will be no value of α that makes Γα true, and so the formula on the right will also turn out to be (vacuously) true.
- QED

Rules of Passage in Translations

RP7: $(\exists\alpha)(\Delta \supset \Gamma\alpha) \rightleftarrows \Delta \supset (\exists\alpha)\Gamma\alpha$
RP8: $(\forall\alpha)(\Delta \supset \Gamma\alpha) \rightleftarrows \Delta \supset (\forall\alpha)\Gamma\alpha$
RP9: $(\exists\alpha)(\Gamma\alpha \supset \Delta) \rightleftarrows (\forall\alpha)\Gamma\alpha \supset \Delta$
RP10: $(\forall\alpha)(\Gamma\alpha \supset \Delta) \rightleftarrows (\exists\alpha)\Gamma\alpha \supset \Delta$

- If anything was damaged, then everyone gets upset.
  - $(\exists x)Dx \supset (\forall x)(Px \supset Ux)$
  - $(\forall x)[Dx \supset (\forall y)(Py \supset Uy)]$ by RP10
- If there are any wildebeest, then if all lions are hungry, they will be eaten.
  - $(\forall x)\{Wx \supset [(\forall y)(Ly \supset Hy) \supset Ex]\}$
  - $(\forall x)\{Wx \supset (\exists y)[(Ly \supset Hy) \supset Ex]\}$ by RP9
  - $(\forall x)(\exists y)\{Wx \supset [(Ly \supset Hy) \supset Ex]\}$ by RP7
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Prenex Normal Form

RP4: $(\forall\alpha)(\Delta \bullet \Gamma\alpha) \rightleftarrows \Delta \bullet (\forall\alpha)\Gamma\alpha$
RP7: $(\exists\alpha)(\Delta \supset \Gamma\alpha) \rightleftarrows \Delta \supset (\exists\alpha)\Gamma\alpha$
RP9: $(\exists\alpha)(\Gamma\alpha \supset \Delta) \rightleftarrows (\forall\alpha)\Gamma\alpha \supset \Delta$
RP10: $(\forall\alpha)(\Gamma\alpha \supset \Delta) \rightleftarrows (\exists\alpha)\Gamma\alpha \supset \Delta$

- Sentences do not have unique PNFs.
- If there is a philosopher whom all philosophers contradict, then there is a philosopher who contradicts himself.
  - $(\exists x)[Fx \bullet (\forall y)(Fy \supset Gyx)] \supset (\exists z)(Fz \bullet Gzz)$
- Transformation #1
  - $(\exists z)(\exists x)\{[Fx \bullet (\forall y)(Fy \supset Gyx)] \supset (Fz \bullet Gzz)\}$ by RP7
  - $(\exists z)(\exists x)\{(\forall y)[Fx \bullet (Fy \supset Gyx)] \supset (Fz \bullet Gzz)\}$ by RP4
  - $(\exists z)(\exists x)(\exists y)\{[Fx \bullet (Fy \supset Gyx)] \supset (Fz \bullet Gzz)\}$ by RP9
- Transformation #2
  - $(\forall x)\{[Fx \bullet (\forall y)(Fy \supset Gyx)] \supset (\exists z)(Fz \bullet Gzz)\}$ by RP10
  - $(\forall x)\{(\forall y)[Fx \bullet (Fy \supset Gyx)] \supset (\exists z)(Fz \bullet Gzz)\}$ by RP4
  - $(\forall x)(\exists y)\{[Fx \bullet (Fy \supset Gyx)] \supset (\exists z)(Fz \bullet Gzz)\}$ by RP9
  - $(\forall x)(\exists y)(\exists z)\{[Fx \bullet (Fy \supset Gyx)] \supset (Fz \bullet Gzz)\}$ by RP7
- The results are in prenex form, and logically equivalent to the original sentence.
- But, they differ in form from each other.

More Translations

1. Everyone loves something. (Px, Lxy)
2. No one knows everything. (Px, Kxy)
3. No one knows everyone.
4. Every woman is stronger than some man. (Wx, Mx, Sxy: x is stronger than y)
5. No cat is smarter than any horse. (Cx, Hx, Sxy: x is smarter than y)
6. Dead men tell no tales. (Dx, Mx, Tx, Txy: x tells y)
7. There is a city between New York and Washington. (Cx, Bxyz: y is between x and z)
8. Everyone gives something to someone. (Px, Gxyz: y gives x to z)
9. A dead lion is more dangerous than a live dog. (Ax: x is alive, Lx, Dx, Dxy: x is more dangerous than y)
10. A lawyer who pleads his own case has a fool for a client. (Lx, Fx, Pxy: x pleads y's case; Cxy: y is a client of x)