

Philosophy 240: Symbolic Logic

Russell Marcus
Hamilton College
Fall 2011

Class 29 - Semantics for Predicate Logic

Theories

- A *theory* is a set of sentences.
- A formal theory is a set of sentences of a formal language.
- We identify a theory by its theorems, the set of sentences provable within that theory.
- Many interesting formal theories are infinite.
 - Rules of inference generate an infinite number of theorems.

Constructing Formal Theories

1. Specify a language
 - vocabulary
 - formation rules for wffs

Syntax for **PL** and **M**

Vocabulary for **PL**

Capital letters A...Z

Five connectives: \sim , \bullet , \vee , \supset , \equiv

Punctuation: $()$, $[]$, $\{ \}$

Formation rules for Wffs of **PL**

1. A single capital English letter is a wff.
2. If α is a wff, so is $\sim\alpha$.
3. If α and β are wffs, then so are:
 $(\alpha \bullet \beta)$
 $(\alpha \vee \beta)$
 $(\alpha \supset \beta)$
 $(\alpha \equiv \beta)$
4. These are the only ways to make wffs.

Vocabulary for **M**

Capital letters A...Z used as one-place predicates

Lower case letters used as singular terms

a, b, c,...u are constants.

v, w, x, y, z are variables.

Five connectives: \sim , \bullet , \vee , \supset , \equiv

Quantifier symbols: \exists , \forall

Punctuation: $()$, $[]$, $\{ \}$

Formation Rules for Wffs of **M**

1. A predicate (capital letter) followed by a constant or variable (lower-case letter) is a wff.
2. For any variable β , if α is a wff that does not contain either ' $(\exists\beta)$ ' or ' $(\forall\beta)$ ', then ' $(\exists\beta)\alpha$ ' and ' $(\forall\beta)\alpha$ ' are wffs.
3. If α is a wff, so is $\sim\alpha$.
4. If α and β are wffs, then so are:
 $(\alpha \bullet \beta)$
 $(\alpha \vee \beta)$
 $(\alpha \supset \beta)$
 $(\alpha \equiv \beta)$
5. These are the only ways to make wffs.

Constructing Formal Theories

1. Specify a language
 - ▶ vocabulary
 - ▶ formation rules for wffs
2. Add formation rules for wffs.

To construct a formal theory, we select some of the wffs as our theorems.

3a. Specify theorems

- ▶ We can list all of our theorems: finite theories
- ▶ We can specify axioms and rules of inference.
- ▶ Proof Theory
 - Euclidean Geometry
 - Newtonian Mechanics
- ▶ Different theories can be written in the same language.

3b. Provide a semantics for the theory.

- ▶ Specify truth conditions and truth values for wffs
- ▶ Model Theory (Truth)

Goodness for Theories

- In some theories, the provable theorems are exactly the same as the true wffs.
 - ▶ *Soundness*: all the provable theorems are true
 - ▶ *Completeness*: all the truths are provable
- In more sophisticated theories, proof separates from truth
 - ▶ Gödel's first incompleteness theorem
 - ▶ There are true sentences that are not provable within any given system.
 - ▶ Model theory and proof theory provide different results.

Semantics and Proof Theory for **PL** and for **M**

- In **PL**, our semantics used truth tables.
 - ▶ Interpretations of **PL**
 - **Step 1. Assign 1 or 0 to each atomic sentence.**
 - Only finitely many ($2^{26} = \sim 6.7$ million) possible interpretations.
 - We could use infinitely many simple terms: $P, P', P'', P''', P'''' \dots$
 - **Step 2. Assign truth values to complex propositions by combining, according to the truth table definitions, the truth values of the atomic sentences.**
 - ▶ In **M**, and the other languages of predicate logic we will study, the semantics are more complicated.
 - interpretation, satisfaction, logical truth, validity
- In proof theory, we construct a system of inference using the formal language we have specified.
 - ▶ In **PL**, our proof system was our eighteen rules of natural deduction.
 - no axioms
 - conditional and indirect proof
 - ▶ Other proof systems use axioms.

Semantics for M

- Separating the syntax of our language from its semantics allows us to treat our formal languages as completely uninterpreted.
 - Intuitively, we know what the logical operators mean.
 - But until we specify a formal interpretation, we are free to interpret them as we wish.
- Our constants and predicates and quantifiers are, as far as the syntax of our language specifies, uninterpreted.
- To look at the logical properties of the language, we construct formal semantics.
- The first step in formal semantics is to show how to provide an interpretation of the language.
- Then, we can determine the logical truths.
 - The wffs that come out as true under every interpretation.

Interpretations of M

- To define an interpretation in M, or in any of its extensions, we have to specify how to handle constants, predicates, and quantifiers.
 - ▶ We use some set theory.
 - ▶ Not in our object language, but in our meta-language.
- **Step 1.** Specify a set to serve as a domain of interpretation, or domain of quantification.
 - ▶ We can consider small finite domains
Domain₁ = {1, 2, 3}
Domain₂ = {Barack Obama, Hillary Clinton, and Rahm Emanuel}.
 - ▶ We can consider larger domains, like a universe of everything.
- **Step 2.** Assign a member of the domain to each constant.
 - a: 1; b: 2; c: 3
 - a: Obama; b: Clinton
- **Step 3.** Assign some set of objects in the domain to each predicate.
 - ▶ 'Ex' may stand for 'x has been elected president'
 - ▶ In Domain₁, the interpretation of 'Ex' will be empty.
 - ▶ In Domain₂, it will be {Barack Obama}.
- **Step 4.** Use the customary truth tables for the interpretation of the connectives.

Satisfaction and Truth-for-an-Interpretation

- Objects in the domain may satisfy predicates.
 - ▶ Ordered n-tuples may satisfy relations.
- A wff will be satisfiable if there are objects in the domain of quantification which satisfy the predicates indicated in the wff.
 - ▶ A universally quantified sentence is satisfied if it is satisfied by all objects in the domain.
 - ▶ An existentially quantified sentence is satisfied if it is satisfied by some object in the domain.
- A wff will be true-for-an-interpretation if all objects in the domain of quantification satisfy the predicates indicated in the wff.
- We call an interpretation on which all of a set of statements come out true a *model*.

An Interpretation of a Theory

- Theory
 1. $Pa \cdot Pb$
 2. $Wa \cdot \sim Wb$
 3. $(\exists x)Px$
 4. $(\forall x)Px$
 5. $(\forall x)(Wx \supset Px)$
 6. $(\forall x)(Px \supset Wx)$
- Step 1: Specify a set to serve as a domain of interpretation, or domain of quantification.
 - ▶ Domain: {Bob Simon, Rick Werner, Katheryn Doran, Todd Franklin, Marianne Janack, Russell Marcus, Martin Shuster}
- Step 2: Assign a member of the domain to each constant.
 - ▶ a: Katheryn Doran
 - ▶ b: Bob Simon
 - Notice: no other constants in our theory
 - Some objects remain without names
- Step 3: Assign some set of objects in the domain to each predicate.
 - ▶ Px : {Bob Simon, Rick Werner, Katheryn Doran, Todd Franklin, Marianne Janack, Russell Marcus, Martin Shuster}
 - ▶ Wx : {Katheryn Doran, Marianne Janack}

Models in M

- We call an interpretation on which all of a set of given statements come out true a *model*.
- Given our interpretations of the predicates, not every sentence in our set is satisfied.
 - ▶ 1-5 are satisfied.
 - ▶ 6 is not.
- If we were to delete sentence 6 from our list, our interpretation would be a model.

1. $Pa \cdot Pb$
2. $Wa \cdot \sim Wb$
3. $(\exists x)Px$
4. $(\forall x)Px$
5. $(\forall x)(Wx \supset Px)$
6. $(\forall x)(Px \supset Wx)$

Domain: {Bob Simon, Rick Werner, Katheryn Doran, Todd Franklin, Marianne Janack, Russell Marcus, Martin Shuster}

a: Katheryn Doran

b: Bob Simon

Px: {Bob Simon, Rick Werner, Katheryn Doran, Todd Franklin, Marianne Janack, Russell Marcus, Martin Shuster}

Wx: {Katheryn Doran, Marianne Janack}

Constructing a Model

- Theory

1. $(\forall x)(Px \supset Qx)$
2. $(\exists x)(Px \cdot Rx)$
3. $(\exists x)(Px \cdot \sim Rx)$
4. $(\exists x)(Qx \cdot \sim Rx)$
5. $(Pa \cdot Pb) \cdot Qc$

- **Step 1.** Specify a set to serve as a domain of interpretation, or domain of quantification.

Domain = {Persons}

- **Step 2.** Assign a member of the domain to each constant.

a = Barack Obama

b = Condoleezza Rice

c = Neytiri (from *Avatar*)

- **Step 3.** Assign some set of objects in the domain to each predicate.

$Px = \{\text{Human Beings}\}$

$Qx = \{\text{Persons}\}$

$Rx = \{\text{Males}\}$

- **Step 4.** Use the customary truth tables for the interpretation of the connectives.

Logical Truth in M

- A wff of M will be logically true if it is true for every interpretation.
- For **PL**, the notion of logical truth was simple.
 - Just look at the truth tables.
- For **M**, and even more so for **F** (full first-order logic), the notion of logical truth is just naturally complicated by the fact that we are analyzing parts of propositions.
- Here are two logical truths of **M**:
 - $(\forall x)(Px \vee \sim Px)$
 - $Pa \vee [(\forall x)Px \supset Qa]$
- We can show that a wff is a theorem (logical truth) proof-theoretically and model-theoretically.

Proof-Theoretic Argument

$$(\forall x)(Px \vee \sim Px)$$

- | | | |
|----|------------------------------------|-------------|
| 1. | $\sim(\forall x)(Px \vee \sim Px)$ | AIP |
| 2. | $(\exists x)\sim(Px \vee \sim Px)$ | 1, QE |
| 3. | $\sim(Pa \vee \sim Pa)$ | 2, EI |
| 4. | $\sim Pa \cdot \sim\sim Pa$ | 3, DM |
| 5. | $(\forall x)(Px \vee \sim Px)$ | 1-4, IP, DN |

Model-Theoretic Argument

$$Pa \vee [(\forall x)Px \supset Qa]$$

- Consider an interpretation on which ‘ $Pa \vee [(\forall x)Px \supset Qa]$ ’ is false.
- The object assigned to ‘ a ’ will not be in the set assigned to ‘ Px ’, and there is some counterexample to ‘ $(\forall x)Px \supset Qa$ ’.
- But, any counter-example to a conditional statement has to have a true antecedent.
- So, every object in the domain will have to be in the set assigned to ‘ Px ’.
 - Tilt
- So, no interpretation will make that sentence false.
- So, ‘ $Pa \vee [(\forall x)Px \supset Qa]$ ’ is logically true.

Another Logical Truth

$$(\exists x)Px \vee (\forall x)(Qx \supset \sim Px)$$

- Try it both ways!

Validity

- A valid argument will have to be valid under any interpretation, using any domain.
- Our proof system has given us ways to show that an argument is valid.
- But when we introduced our system of inference for **PL**, we already had a way of distinguishing the valid from the invalid arguments, using truth tables.
- In **M**, we need a corresponding method for showing that an argument is invalid.
- An invalid argument will have counter-examples, interpretations on which the premises come out true and the conclusion comes out false.

Coming Up

- Monday
 - Invalid Arguments in Predicate Logic
 - Constructing Counterexamples
- Friday
 - Philosophy Friday
 - Existential Quantifiers and Ontological Commitments