Philsoophy 240: Symbolic Logic
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Axiom Systems

I. Hunter's System PS

Hunter's System PS, in *Metalogic*, is a Hilbert-style axiom system for propositional logic, with three axiom schemata:

> PS1: $\alpha \supset (\beta \supset \alpha)$
> PS2: $(\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$
> PS3: $(\sim\alpha \supset \sim\beta) \supset (\beta \supset \alpha)$

Hunter also takes one rule of inference, modus ponens.

> MP: If α and β are formulas, β is a consequence in system PS of α and $\alpha \supset \beta$.
> i.e. $\alpha, (\alpha \supset \beta) \vdash_{PS} \beta$

Hunter's PS is used with a language relevantly similar to our language PS.
Any formula of the language PS of any of the forms PS1, PS2, or PS3 is an axiom of PS.
Thus, there are infinitely many axioms of PS.
An alternative to using axiom schemata would be to take three axioms, and an extra rule of inference, viz.

> PS1*: $P \supset (Q \supset P)$
> PS2*: $(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$
> PS3*: $(\sim P \supset \sim Q) \supset (Q \supset P)$

> MP: $\alpha, (\alpha \supset \beta) \vdash_{PS} \beta$
> Substitution: For any propositional symbol α, and any wff β appearing in wff α, If $\vdash_{PS} \alpha$, then $\vdash_{PS}$ $\forall_{\alpha}^{\beta}$, where 'A_{α}^{β}' is the result of substituting 'α' for 'β' throughout.

Hunter's axiomatic system is provably equivalent to our more-familiar natural deduction systems.
There is a metalogical result called the deduction theorem which allows us to move between the two kinds of systems.
Frege's original *Begriffsschrift* used a Hilbert-style axiomatic system, with a different axiomatization.
See the appendices in Richard Mendelsohn's *The Philosophy of Gottlob Frege*, for a translation of the *Begriffsschrift* into modern notation.
Natural deduction systems like the one in *What Follows* are due largely to Gerhard Gentzen's work in the 1930s and 1940s.
There is a paper on the history of natural deduction by Pelletier which might be worth a look for a term paper.
Both Frege's axiomatization and the standard systems of natural deduction are complete which means that all logical truths are provable within the system.
One's choice of system is thus arbitrary among the various complete systems.

II. Sample Derivations in PS

Example A1: $\vdash_{PS}$ P ⊃ P

1. P ⊃ ((P ⊃ P) ⊃ P)	PS1
2. (P ⊃ ((P ⊃ P) ⊃ P)) ⊃ ((P ⊃ (P ⊃ P)) ⊃ (P ⊃ P))	PS2
3. (P ⊃ (P ⊃ P)) ⊃ (P ⊃ P)	MP, 1, 2
4. P ⊃ (P ⊃ P)	PS1
5. P ⊃ P	3, 4, MP

QED

Note that we could apply the same technique to derive any formula of the form ⌈A ⊃ A⌉.
That is, by substituting '(P ⊃ ~Q) ⊃ ~Q for 'P' in the above proof, we can construct a proof of '((P ⊃ ~Q) ⊃ ~P) ⊃ ((P ⊃ ~Q) ⊃ ~P)'.
Once a theorem has been established, it is fair to use it in the proofs of further theorems.
For example:

Example A2: $\vdash_{PS}$ ~P ⊃ (P ⊃ P)

1. P ⊃ P	Ex A1
2. (P ⊃ P) ⊃ (~P ⊃ (P ⊃ P))	PS 1
3. ~P ⊃ (P ⊃ P)	MP, 1, 2

QED

The next two pages contain longer proofs in PS.

$\vdash_{PS}$ $(P \supset (Q \supset R)) \supset (Q \supset (P \supset R))$

1. $Q \supset (P \supset Q)$	Ax 1
2. $(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$	Ax 2
3. $[(P \supset Q) \supset (P \supset R)] \supset [Q \supset ((P \supset Q) \supset (P \supset R))]$	Ax 1
4. $[((P \supset Q) \supset (P \supset Q')) \supset (Q \supset ((P \supset Q) \supset (P \supset R)))] \supset$	
$\{(P \supset (Q \supset R)) \supset [((P \supset Q) \supset (P \supset R)) \supset (Q \supset ((P \supset Q) \supset (P \supset R)))]\}$	Ax 1
5. $(P \supset (Q \supset R)) \supset [((P \supset Q) \supset (P \supset R)) \supset (Q \supset ((P \supset Q) \supset (P \supset R)))]$	4, 3 ⊃E
6. $\{(P \supset (Q \supset R)) \supset [((P \supset Q) \supset (P \supset R)) \supset (Q \supset ((P \supset Q) \supset (P \supset R)))]\} \supset$	
$\{[(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))] \supset [(P \supset (Q \supset R)) \supset (Q \supset ((P \supset Q) \supset (P \supset R)))]\}$	Ax 2
7. $[(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))] \supset [(P \supset (Q \supset R)) \supset (Q \supset ((P \supset Q) \supset (P \supset R)))]$	6, 5, ⊃E
8. $(P \supset (Q \supset R)) \supset [Q \supset ((P \supset Q) \supset (P \supset R))]$	7, 2, ⊃E
9. $[Q \supset ((P \supset Q) \supset (P \supset R))] \supset [(Q \supset (P \supset Q)) \supset (Q \supset (P \supset R))]$	Ax 2
10. $[(Q \supset ((P \supset Q) \supset (P \supset R))) \supset ((Q \supset (P \supset Q)) \supset (Q \supset (P \supset R)))] \supset$	
$\{(P \supset (Q \supset R)) \supset [(Q \supset ((P \supset Q) \supset (P \supset R))) \supset ((Q \supset (P \supset Q)) \supset (Q \supset (P \supset R)))]\}$	Ax 1
11. $(P \supset (Q \supset R)) \supset [(Q \supset ((P \supset Q) \supset (P \supset R))) \supset ((Q \supset (P \supset Q)) \supset (Q \supset (P \supset R)))]$	10, 9, ⊃E
12. $\{(P \supset (Q \supset R)) \supset [(Q \supset ((P \supset Q) \supset (P \supset R))) \supset ((Q \supset (P \supset Q)) \supset (Q \supset (P \supset R)))]\} \supset$	
$\{[(P \supset (Q \supset R)) \supset (Q \supset ((P \supset Q) \supset (P \supset R)))] \supset$	
$[(P \supset (Q \supset R)) \supset ((Q \supset (P \supset Q)) \supset (Q \supset (P \supset R)))]\}$	Ax 2
13. $[(P \supset (Q \supset R)) \supset (Q \supset ((P \supset Q) \supset (P \supset R)))] \supset$	
$[(P \supset (Q \supset R)) \supset ((Q \supset (P \supset Q)) \supset (Q \supset (P \supset R)))]$	12, 11, ⊃E
14. $(P \supset (Q \supset R)) \supset [(Q \supset (P \supset Q)) \supset (Q \supset (P \supset R))]$	13, 8, ⊃E
15. $\{(P \supset (Q \supset R)) \supset [(Q \supset (P \supset Q)) \supset (Q \supset (P \supset R))]\} \supset$	
$\{[(P \supset (Q \supset R)) \supset (Q \supset (P \supset Q))] \supset [(P \supset (Q \supset R)) \supset (Q \supset (P \supset R))]\}$	Ax 2
16. $[(P \supset (Q \supset R)) \supset (Q \supset (P \supset Q))] \supset [(P \supset (Q \supset R)) \supset (Q \supset (P \supset R))]$	15, 14, ⊃E
17. $(Q \supset (P \supset Q)) \supset [(P \supset (Q \supset R)) \supset (Q \supset (P \supset Q))]$	Ax 1
18. $(P \supset (Q \supset R)) \supset (Q \supset (P \supset Q))$	17, 1, ⊃E
19. $(P \supset (Q \supset R)) \supset (Q \supset (P \supset R))$	16, 18, ⊃E

QED

⊢ (P ⊃ Q) ⊃ [(Q ⊃ R) ⊃ (P ⊃ R)]

1. (P ⊃ (Q ⊃ R)) ⊃ ((P ⊃ Q) ⊃ (P ⊃ R))	Ax 2
2. [(P ⊃ (Q ⊃ R)) ⊃ ((P ⊃ Q) ⊃ (P ⊃ R))] ⊃ {(Q ⊃ R) ⊃ [(P ⊃ (Q ⊃ R)) ⊃ ((P ⊃ Q) ⊃ (P ⊃ R))]}	Ax 1
3. (Q ⊃ R) ⊃ [(P ⊃ (Q ⊃ R)) ⊃ ((P ⊃ Q) ⊃ (P ⊃ R))]	2, 1, ⊃E
4. {(Q ⊃ R) ⊃ [(P ⊃ (Q ⊃ R)) ⊃ ((P ⊃ Q) ⊃ (P ⊃ R))]} ⊃ {[(Q ⊃ R) ⊃ (P ⊃ (Q ⊃ R))] ⊃ [(Q ⊃ R) ⊃ ((P ⊃ Q) ⊃ (P ⊃ R))]}	Ax 2
5. [(Q ⊃ R) ⊃ (P ⊃ (Q ⊃ R))] ⊃ [(Q ⊃ R) ⊃ ((P ⊃ Q) ⊃ (P ⊃ R))]	4, 3, ⊃E
6. (Q ⊃ R) ⊃ (P ⊃ (Q ⊃ R))	Ax 1
7. (Q ⊃ R) ⊃ ((P ⊃ Q) ⊃ (P ⊃ R))	5, 6, ⊃E
8. [(Q ⊃ R) ⊃ ((P ⊃ Q) ⊃ (P ⊃ R))] ⊃ {[(Q ⊃ R) ⊃ (P ⊃ Q)] ⊃ [(Q ⊃ R) ⊃ (P ⊃ R)]}	Ax 2
9. [(Q ⊃ R) ⊃ (P ⊃ Q)] ⊃ [(Q ⊃ R) ⊃ (P ⊃ R)]	8, 7, ⊃E
10. [((Q ⊃ R) ⊃ (P ⊃ Q)) ⊃ ((Q ⊃ R) ⊃ (P ⊃ R))] ⊃ {(P ⊃ Q) ⊃ [((Q ⊃ R) ⊃ (P ⊃ Q)) ⊃ ((Q ⊃ R) ⊃ (P ⊃ R))]}	Ax 1
11. (P ⊃ Q) ⊃ [((Q ⊃ R) ⊃ (P ⊃ Q)) ⊃ ((Q ⊃ R) ⊃ (P ⊃ R))]	10, 9, ⊃E
12. {(P ⊃ Q) ⊃ [((Q ⊃ R) ⊃ (P ⊃ Q)) ⊃ ((Q ⊃ R) ⊃ (P ⊃ R))]} ⊃ {[(P ⊃ Q) ⊃ ((Q ⊃ R) ⊃ (P ⊃ Q))] ⊃ [(P ⊃ Q) ⊃ ((Q ⊃ R) ⊃ (P ⊃ R))]}	Ax 2
13. {(P ⊃ Q) ⊃ [(Q ⊃ R) ⊃ (P ⊃ Q)]} ⊃ {(P ⊃ Q) ⊃ [(Q ⊃ R) ⊃ (P ⊃ R)]}	12, 11, ⊃E
14. (P ⊃ Q) ⊃ [(Q ⊃ R) ⊃ (P ⊃ Q)]	Ax 1
15. (P ⊃ Q) ⊃ [(Q ⊃ R) ⊃ (P ⊃ R)]	13, 14, ⊃E

QED