Philosophy 240: Symbolic Logic
Fall 2010
Mondays, Wednesdays, Fridays: 9am - 9:50am

Hamilton College
Russell Marcus
rmarcus1@hamilton.edu

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Rules of Implication I[1] (§7.1)

I. A System of Natural Deduction

We have used truth tables, and the short-cut method, to separate the valid arguments from the invalid arguments, according to a semantic definition of validity: an argument is valid if there are no substitutions of the propositional variables on which the premises come out true and the conclusion comes out false.
But, the truth-table method gets increasingly and prohibitively arduous as the complexity of an argument grows.
And, the short-cut method, while amusing, requires ingenuity and can be just as long as the truth-table method.
More importantly, while semantic tests of validity are completely effective in propositional logic, in more advanced logical systems they do not suffice to determine all the valid arguments.
So, when we get to predicate logic, we will require another method.
That other method, called natural deduction (and sometimes just called proofs), is best learned at this point.
A proof (or deduction or derivation) is a sequence of wffs every member of which either is an assumption or follows from earlier wffs in the sequence according to specified rules.

For natural deductions, we use the language of propositional logic, plus 8 Rules of Implication and 10 Rules of Replacement.
These rules are chosen so that they are complete: every valid argument will be provable using our rules.
But, they are also chosen arbitrarily, in the sense that there are infinitely many systems of rules which are complete.
One can devise a system of logic with very few rules, though the resulting proofs become very long.
One can devise a system of logic so that one's proofs become very short, but the required number of rules can be impossibly large.
So, we choose a moderate number of rules (18) such that there are not too many to memorize and the proofs are not too long.

The rules we choose are defined purely syntactically, in terms of their form.
They must preserve truth: given true premises, the rules must never yield a false conclusion.
A rule preserves truth if every substitution instance, every argument of that form, is valid.
We can prove that each of the rules preserves truth using the indirect truth table method.

Deductions generally begin with any number of premises, and end with a conclusion.
A deduction is valid if every step is either a premise or derived from premises or previous steps using valid rules of implication.

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[1] Hurley calls these rules implication rules. They are better thought of as rules of inference, but even that term is awkward I will use Hurley's terminology for the purposes of consistency with his text.

II. Some Rules of Implication

Consider the validity of each of the following arguments:

1.	$A \supset B$	
	A	$/ B$
2.	$(E \cdot I) \supset D$	
	$(E \cdot I)$	$/ D$
3.	$\sim G \supset (F \cdot H)$	
	$\sim G$	$/ F \cdot H$

Note that they share their common (valid) form:

$$\begin{array}{lll} \alpha \supset \beta & & \\ \alpha & / \beta & \text{Modus Ponens (MP)} \end{array}$$

This form is called *Modus Ponens*, and abbreviated MP.
For example, "If I own a Toyota, then I own a car. I own a Toyota. So, I own a car."
Note that we can substitute simple or complex formulae for α and β.

Here is another example of MP:

$$\begin{array}{ll} [(H \vee G) \supset I] \supset (K \cdot \sim L) & \\ [(H \vee G) \supset I] & / (K \cdot \sim L) \end{array}$$

The following three forms are also valid.
We can check them, using indirect truth tables.

$$\begin{array}{lll} \alpha \supset \beta & & \\ \sim\beta & / \sim\alpha & \text{Modus Tollens (MT)} \end{array}$$

This form is called *Modus Tollens*, abbreviated MT.
For example, "If I own a Toyota, I own a car. I don't own a car. So, I don't own a Toyota."

$$\begin{array}{lll} \alpha \vee \beta & & \\ \sim\alpha & / \beta & \text{Disjunctive Syllogism (DS)} \end{array}$$

This form is called *Disjunctive Syllogism*, abbreviated DS.
For example, "I will have soup or salad. I don't have soup. So, I will have salad."

$$\begin{array}{lll} \alpha \supset \beta & & \\ \beta \supset \gamma & / \alpha \supset \gamma & \text{Hypothetical Syllogism (HS)} \end{array}$$

This form is called *Hypothetical Syllogism*, abbreviated HS.
For example, "If I own a Toyota I own a car. If I own a car, I have to pay for insurance. So, If I own a Toyota, I have to pay for insurance."

The following two forms are invalid.
Again, we can check them, using truth tables, or indirect truth tables.

$\alpha \supset \beta$
$\beta \quad / \alpha$

This form is called the *Fallacy of Affirming the Consequent*.
For example, “If I own a Toyota, I own a car. I own a car. So I own a Toyota.”
Note that the premises may be true while the conclusion is false.

$\alpha \supset \beta$
$\sim\alpha \quad / \sim\beta$

This form is called the *Fallacy of Denying the Antecedent*.
For example, “If I own a Toyota, I own a car. I don’t own a Toyota. So, I don’t own a car.

III. Examples of deductions

The following derivation will prove that the argument (presented in the first four premises and the conclusion) is valid.

1. $(X \supset Y) \supset T$		
2. $S \vee \sim T$		
3. $U \supset \sim S$		
4. U	$/ \sim(X \supset Y)$	
5. $\sim S$	3, 4, MP	(taking ‘U’ for P and ‘~S’ for Q)
6. $\sim T$	2, 5, DS	(taking ‘S’ for P and ‘~T’ for Q)
7. $\sim(X \supset Y)$	1, 6, MT	(taking ‘X ⊃ Y’ for P and ‘T’ for Q)

QED

Notes on the above deduction:

All lines except the premises require justification.
Justifications include the lines and rule of implication used to generate the new conclusion. For example, ‘3, 4, MP’ means that the current line is derived directly from lines 3 and 4 by a use of the rule of Modus Ponens.
The conclusion, written after a single slash following the last premise is not technically part of the deduction.
Deductions are sometimes called proofs, and sometimes called derivations.
The explanations such as “taking ‘U’ for P and ‘~S’ for Q” are not required elements of the derivation.
‘QED’ stands for ‘Quod erat demonstratum’, meaning ‘Thus it has been shown’, and serves as a logician’s punctuation mark: “I’m done!” It is not required, but looks neat.

Another example:

1. $\sim G \supset [G \vee (S \supset A)]$	
2. $(S \vee L) \supset \sim G$	
3. $S \vee L$	
4. $A \supset G$	/ L
5. $\sim G$	2, 3, MP
6. $G \vee (S \supset A)$	1, 5, MP
7. $S \supset A$	6, 5, DS
8. $S \supset G$	7, 4, HS
9. $\sim S$	8, 5, MT
10. L	3, 9, DS

QED

Some hints for constructing a derivation:

Start with simple sentences, or negations of simple negations.
Plan ahead, work backwards on the side.
Don't worry about extraneous lines: not every line must be used.
Some lines may be used more than once.

IV. Exercises. Derive the conclusions of each of the following arguments using natural deduction.

1.
1. $(A \cdot B) \supset (E \vee D)$
2. $A \cdot B$
3. $\sim E$ / D

2.
1. $\sim D \vee (H \vee F)$
2. $H \supset G$
3. $\sim\sim D$
4. $\sim G$ / F

3.
1. $X \supset Y$
2. $\sim Z$
3. $Y \supset Z$
4. $X \vee W$ / W

4.
1. $A \supset \sim B$
2. $A \vee (D \supset E)$
3. $\sim B \supset E$
4. $\sim E$ / $\sim D$

Solutions may vary.