

Class 8 - September 13
Indirect Truth Tables for Invalidity and Inconsistency (§6.5)

I. Invalidity by counter-example method (natural interpretation)

An argument is invalid if it possible to derive a false (or untrue) conclusion from true premises. One way to show that an argument is invalid is to present an argument of the same form with clearly true premises and a clearly false conclusion.

So, for example, consider the fallacy of affirming the consequent:

If it is raining, then Mary has an umbrella.
Mary has an umbrella.
So, it is raining.

We can regiment the argument in our language as:

$$\begin{array}{l} P \supset Q \\ Q \quad / P \end{array}$$

If the original argument were valid, then there would be no possible values for 'P' and 'Q' on which the premises were true and the conclusion were false.

To show that the argument is invalid, we construct a counter-example by substituting propositions that make the premises true and conclusion false.

For example, take:

P: All people are women.
Q: Erin is a woman.

It is clearly the case that if all people are women, then Erin is a woman.

It is also clear that Erin is a woman.

But, it is not the case that all people are women.

Here is another invalid argument:

It's not the case that it is both raining and chilly.
Either it is chilly or it is raining.
So, it is chilly.

On any given day, the premises might be true, and the conclusion might be true.

But it is also possible for the premises to be true and the conclusion false.

We can represent the argument in our language:

$$\begin{array}{l} \sim(R \cdot C) \\ C \vee R \quad / C \end{array}$$

Now, we can construct a counter-example:

- R: Obama is president.
- C: $2+2=5$

Since it is possible to construct a counter-example, the original argument form is invalid.

II. Limitations of Truth Tables

The natural interpretation method becomes difficult for any complex argument. Furthermore, we have a mechanical procedure for determining counter-examples. We can use the truth table method for any argument, to determine its validity. But the method becomes unwieldy as the number of variables in an argument grows. With merely 5 variables, for example, the truth table would be 32 lines. The truth table for an argument which contained 10 propositional variables would require 1024 lines. Fortunately, there's a shortcut method.

To show that an argument is valid, you have to show that *every row* of the truth table doesn't have true premises and a false conclusion.
 How many rows do we need to show that an argument is invalid?
 Only one! - the counter-example

III. Indirect Truth Tables for Arguments

To determine whether an argument is valid or invalid, construct a counter-example:

1. Assign values to make the conclusion false.
2. Try to make all premises true.
3. If 1 and 2 are possible, then the argument is invalid.
 Specify the counter-example.
4. If 1 and 2 are impossible, then the argument is valid.
 If there is a counterexample, this method will be able to find it.

Note: Steps 1 and 2 can be done in any order!

Consider the argument:

1. $G \equiv H$
2. $G \quad / \quad \sim H$

First, write it out, as you would a normal truth table for an argument:

G	\equiv	H	/	G	//	\sim	H

Next, assign the value 'true' to H, in order to make the conclusion false:

G	≡	H	/	G	//	~	H
						⊥	⊤

Carry this value over to any other H in the argument:

G	≡	H	/	G	//	~	H
		⊤				⊥	⊤

Now, assign a value to G which makes the premises true:

G	≡	H	/	G	//	~	H
⊤	⊤	⊤		⊤		⊥	⊤

The argument is invalid when G is true and H is true.

Consider:

1. $C \supset (D \supset E)$
2. $D \supset (E \supset F)$ / $C \supset (D \supset F)$

The only way to make the conclusion false is to assign true to C and to D, and false to F:

C	⊃	(D	⊃	E)	/	D	⊃	(E	⊃	F)	//	C	⊃	(D	⊃	F)
												⊤	⊥	⊤	⊥	⊥

Carry these values over to the premises:

C	⊃	(D	⊃	E)	/	D	⊃	(E	⊃	F)	//	C	⊃	(D	⊃	F)
⊤		⊤				⊤				⊥		⊤	⊥	⊤	⊥	⊥

In order to make the first premise true, E must also be true:

C	⊃	(D	⊃	E)	/	D	⊃	(E	⊃	F)	//	C	⊃	(D	⊃	F)
⊤	⊤	⊤	⊤	⊤		⊤		⊤		⊥		⊤	⊥	⊤	⊥	⊥

But now the second premise is made false.

If we tried to make the second premise true, by making E false, the first premise would come out false.

There was no other way to make the conclusion false, so there is no counter-example.

This argument is thus valid.

In some arguments, there is more than one way to make a conclusion false, or to make premises true.

You may have to try more than one.

Once you arrive at a counter-example, you may stop.

Some arguments have multiple counter-examples, but you only need one to demonstrate the invalidity of an argument.

Consider:

1. $I \supset K$
2. $K \supset J$ $/ I \cdot J$

Note that there are three ways to make the conclusion false.

I	\supset	K	/	K	\supset	J	//	I	\cdot	J
								T	\perp	\perp
								\perp	\perp	T
								\perp	\perp	\perp

We'll try them in order.

On the first assignment, there is no way to assign a truth value to K which makes the premises true.

I	\supset	K	/	K	\supset	J	//	I	\cdot	J
T		?		?		\perp		T	\perp	\perp
								\perp	\perp	T
								\perp	\perp	\perp

We'll have to move on to the second option.

I	\supset	K	/	K	\supset	J	//	I	\cdot	J
								T	\perp	\perp
\perp						T		\perp	\perp	T
								\perp	\perp	\perp

Here, we can assign either value to K, and still have a counter-example.

This argument is invalid when I is false, J is true, and K is either true or false.

Since we found a counter-example in the second option, there is no need to continue with the third option.

Consider a slightly longer one:

1. $T \supset (U \vee X)$
2. $U \supset (Y \vee Z)$
3. $Z \supset A$
4. $\sim(A \vee Y) \quad / \sim T$

Start with the conclusion, making T true in order to make its negation false, carrying that assignment into the first premise:

T	\supset	(U	\vee	X)	/	U	\supset	(Y	\vee	Z)	/	Z	\supset	A	/	\sim	(A	\vee	Y)	//	\sim	T	
T																						\perp	T

From the first premise, 'U \vee X' must be true, but there are three ways to assign values to make it so. Similarly, there are multiple ways to assign values for the second and third premises.

But in the fourth premise, there is only one possible assignment:

T	\supset	(U	\vee	X)	/	U	\supset	(Y	\vee	Z)	/	Z	\supset	A	/	\sim	(A	\vee	Y)	//	\sim	T	
T																T	\perp	\perp	\perp			\perp	T

Carrying these assignments over:

T	\supset	(U	\vee	X)	/	U	\supset	(Y	\vee	Z)	/	Z	\supset	A	/	\sim	(A	\vee	Y)	//	\sim	T	
T								\perp						\perp		T	\perp	\perp	\perp			\perp	T

Now, we know, by inspecting the third premise, that Z must also be false.

T	\supset	(U	\vee	X)	/	U	\supset	(Y	\vee	Z)	/	Z	\supset	A	/	\sim	(A	\vee	Y)	//	\sim	T	
T								\perp				\perp	\perp	\perp		T	\perp	\perp	\perp			\perp	T

Carrying this assignment over, 'Y \vee Z' has been made false, and so U must be made false, also:

T	\supset	(U	\vee	X)	/	U	\supset	(Y	\vee	Z)	/	Z	\supset	A	/	\sim	(A	\vee	Y)	//	\sim	T	
T						\perp	T	\perp	\perp	\perp		\perp	\perp	\perp		T	\perp	\perp	\perp			\perp	T

Carrying the value of U to the first premise:

T	\supset	(U	\vee	X)	/	U	\supset	(Y	\vee	Z)	/	Z	\supset	A	/	\sim	(A	\vee	Y)	//	\sim	T	
T		\perp				\perp	T	\perp	\perp	\perp		\perp	\perp	\perp		T	\perp	\perp	\perp			\perp	T

We now are constrained to make X true, in order to make the first premise true.

T	\supset	(U	\vee	X)	/	U	\supset	(Y	\vee	Z)	/	Z	\supset	A	/	\sim	(A	\vee	Y)	//	\sim	T	
T	T	\perp	T	T		\perp	T	\perp	\perp	\perp		\perp	\perp	\perp		T	\perp	\perp	\perp			\perp	T

The counterexample is constructed. The argument is shown invalid when: T and X are true; and U, Y, Z, and A are all false.

IV. Exercises A. Determine whether each of the following arguments is valid. If invalid, specify a counter-example.

1. 1. $(F \cdot G) \equiv H$
 2. $\sim H$ / $\sim G$

2. 1. $P \supset Q$
 2. $Q \supset P$ / $P \equiv Q$

3. 1. $\sim(P \cdot Q)$
 2. $(\sim P \cdot \sim Q) \supset (R \cdot S)$
 3. $S \supset R$ / R

4. 1. $A \supset (B \supset G)$
 2. $D \supset (E \supset F)$
 3. $B \supset (D \cdot E)$
 4. $B \vee F$ / $A \equiv G$

V. Using indirect truth tables to determine whether a set of propositions is consistent

Two or more propositions are consistent if there is at least one line of the truth table in which they are all true.

That is, if we can find an assignment of truth values to component propositions which makes all the main connectives of the propositions in a set true, then we have shown them consistent.

This assignment will be called a consistent valuation.

In no consistent valuation is possible, then the set is inconsistent.

The method:

To determine whether a set of propositions is consistent:

1. Assign values to propositional variables to make each statement true.
2. If this is possible, then the set is consistent.
 Provide a consistent valuation.
3. If it is not possible, then the set is inconsistent.

Consider the following set of propositions:

- $(A \cdot B) \supset F$
- $B \supset (D \vee \sim E)$
- $F \supset (E \vee \sim D)$
- $A \cdot E$

To determine if they are consistent, line them up, just as we line up the arguments.

The only difference is that a set of sentences has no conclusion.

We only use single slashes between the propositions.

(A	·	B)	⊃	F	/	B	⊃	(D	∨	~	E)	/	F	⊃	(E	∨	~	D)	/	A	·	E	

Let's start with the last proposition, since there is only one way to make it true:

(A	·	B)	⊃	F	/	B	⊃	(D	∨	~	E)	/	F	⊃	(E	∨	~	D)	/	A	·	E	
																					⊤	⊤	⊤

Carry those values through the rest of the set, and evaluate the negation of 'E' in the second proposition:

(A	·	B)	⊃	F	/	B	⊃	(D	∨	~	E)	/	F	⊃	(E	∨	~	D)	/	A	·	E
⊤										⊥	⊤				⊤					⊤	⊤	⊤

There are no other obvious, forced moves.

The antecedent in the first proposition may be either true or false, depending on the value given to B.

The consequent in the second may be either true or false, depending on the value given to D.

The consequent in the third proposition, and thus the whole third proposition, will be true, but that doesn't determine the values of either F or D:

(A	·	B)	⊃	F	/	B	⊃	(D	∨	~	E)	/	F	⊃	(E	∨	~	D)	/	A	·	E
⊤										⊥	⊤			⊤	⊤	⊤				⊤	⊤	⊤

We must arbitrarily choose a next place to work.

I'll choose to start with B; we may assign either a ⊤ or a ⊥.

(A	·	B)	⊃	F	/	B	⊃	(D	∨	~	E)	/	F	⊃	(E	∨	~	D)	/	A	·	E
⊤		⊤				⊤				⊥	⊤			⊤	⊤	⊤				⊤	⊤	⊤
⊤		⊥				⊥				⊥	⊤			⊤	⊤	⊤				⊤	⊤	⊤

We'll try the first line, first.

Assigning ⊤ to B forces us, in the first proposition to assign ⊤ to F.

(A	·	B)	⊃	F	/	B	⊃	(D	∨	~	E)	/	F	⊃	(E	∨	~	D)	/	A	·	E
⊤	⊤	⊤	⊤	⊤		⊤				⊥	⊤		⊤	⊤	⊤	⊤				⊤	⊤	⊤
⊤		⊥				⊥				⊥	⊤			⊤	⊤	⊤				⊤	⊤	⊤

Then, in the second proposition, we must assign ⊤ to D:

(A	·	B)	⊃	F	/	B	⊃	(D	∨	~	E)	/	F	⊃	(E	∨	~	D)	/	A	·	E
⊤	⊤	⊤	⊤	⊤		⊤	⊤	⊤	⊤	⊥	⊤		⊤	⊤	⊤	⊤	⊥	⊤		⊤	⊤	⊤
⊤		⊥				⊥				⊥	⊤			⊤	⊤	⊤				⊤	⊤	⊤

We have thus found a consistent valuation.

We need not return to the problem, and try assigning ⊥ to B.

The set of propositions is consistent when all component variables are assigned ⊤.

There are other possible consistent valuations.

Summary:

For arguments: Stop when you've shown the argument invalid by generating a counter-example.

For sets of sentences: Stop when you've gotten a consistent valuation.

Otherwise, work until you've exhausted all possibilities.

VI. Exercises B. Determine, for each given set of propositions, whether it is consistent. If it is, provide a consistent valuation.

1. $A \supset B$
 $B \supset F$
 $F \supset D$
 $D \supset E$
 $E \supset A$

2. $A \supset (B \cdot G)$
 $B \supset (F \supset E)$
 $G \supset (E \supset \sim F)$
 $A \cdot F$

3. $A \equiv B$
 $(B \vee \sim A) \supset F$
 $(A \vee \sim B) \supset D$
 $D \supset E$
 $\sim F \vee \sim E$

4. $(A \cdot B) \supset F$
 $(D \cdot F) \supset E$
 $E \supset B$
 $A \supset D$
 $\sim A \vee \sim B$

VII. Solutions

Answers to Exercises A:

1. Invalid when F is false, G is true, H is false.
2. Valid
3. Invalid when P is true, Q is false, R is false, S is false; or when P is false, Q is true, R is false, S is false.
4. Invalid. There are quite a few counter-examples. Here are nine:

A	G	B	D	E	F
⊤	⊥	⊥	⊤	⊤	⊤
⊤	⊥	⊥	⊤	⊥	⊤
⊤	⊥	⊥	⊥	⊤	⊤
⊤	⊥	⊥	⊥	⊥	⊤
⊥	⊤	⊤	⊤	⊤	⊤
⊥	⊤	⊥	⊤	⊤	⊤
⊥	⊤	⊥	⊤	⊥	⊤
⊥	⊤	⊥	⊥	⊤	⊤
⊥	⊤	⊥	⊥	⊥	⊤

Answers to Exercises B:

1. Consistent, when all variables are true or all are false.
2. Inconsistent
3. Inconsistent
4. Consistent. There are at least 8 possible consistent valuations.