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Truth Tables for Propositions (§6.3)

I. Truth tables

When we are given a complex proposition, and we know the truth values of the component propositions, we can calculate the truth value of the longer statement.
When we are given a complex proposition, and at least some of the truth values of the component propositions are unknown, the best we can do, at times, is describe how the truth value of the whole varies with the truth value of its parts.

A truth table is a method which can help us characterize any logically complex proposition on the basis of the truth conditions of its component propositions.
Truth tables show us the distributions of all possible truth values of component propositions.
We can construct truth tables for any proposition, using the basic truth tables.
We can also use them to separate valid from invalid arguments.

II. Constructing truth tables for propositions

The Method:

Step 1. How many rows do we need?
1 variable: 2 rows
2 variables: 4 rows
3 variables: 8 rows
4 variables: 16 rows
n variables: 2^n rows

Step 2. Assign truth values to the component variables.
We start truth tables always in the same ways.
See below for examples.

Step 3. Work inside out, placing the column for each letter or connective directly beneath the letter or connective, until you complete the column under the main connective.

Examples:

For an example of a two-row truth table, consider the truth table for ‘P ⊃ P’

P	⊃	P
⊤	⊤	⊤
⊥	⊤	⊥

For an example of a four-row truth table, consider: ‘(P ∨ ~Q) · (Q ⊃ P)’
Step 1: We have two variables, so we need four rows.
Step 2: Assign truth values to component variables:

(P	∨	~	Q)	·	(Q	⊃	P)
⊤			⊤		⊤		⊤
⊤			⊥		⊥		⊤
⊥			⊤		⊤		⊥
⊥			⊥		⊥		⊥

Note that the same values we assign to P in the first column, we also use for P in the last column, and similarly for Q.
Also, all four row truth tables begin with this set of assignments.

Step 3, in stages:
First do the negation:

(P	∨	~	Q)	·	(Q	⊃	P)
⊤		⊥	⊤		⊤		⊤
⊤		⊤	⊥		⊥		⊤
⊥		⊥	⊤		⊤		⊥
⊥		⊤	⊥		⊥		⊥

Then the disjunction and conditional:

(P	∨	~	Q)	·	(Q	⊃	P)
⊤	⊤	⊥	⊤		⊤	⊤	⊤
⊤	⊤	⊤	⊥		⊥	⊤	⊤
⊥	⊥	⊥	⊤		⊤	⊥	⊥
⊥	⊤	⊤	⊥		⊥	⊤	⊥

Finally, the main connective, the conjunction, using the columns for the disjunction and the conditional:

(P	∨	~	Q)	·	(Q	⊃	P)
⊤	⊤	⊥	⊤	⊤	⊤	⊤	⊤
⊤	⊤	⊤	⊥	⊤	⊥	⊤	⊤
⊥	⊥	⊥	⊤	⊥	⊤	⊥	⊥
⊥	⊤	⊤	⊥	⊤	⊥	⊤	⊥

Thus, this proposition is false when P is false and Q is true, and true otherwise.
Note that you only have to write out the truth table once, like the last one in this demonstration.

Here is the start to an eight-line truth table, which we will complete later:

[(P	⊃	Q)	·	(Q	⊃	R)]	⊃	(P	⊃	R)
⊤		⊤		⊤		⊤		⊤		⊤
⊤		⊤		⊤		⊥		⊤		⊥
⊤		⊥		⊥		⊤		⊤		⊤
⊤		⊥		⊥		⊥		⊤		⊥
⊥		⊤		⊤		⊤		⊥		⊤
⊥		⊤		⊤		⊥		⊥		⊥
⊥		⊥		⊥		⊤		⊥		⊤
⊥		⊥		⊥		⊥		⊥		⊥

Note that the columns under each instance of the same variable are identical.
In general, to construct a truth table:

> The first variable is assigned ⊤ in the top half and assigned ⊥ in the bottom half.
> The second variable is assigned ⊤ in the top quarter, ⊥ in the second quarter, ⊤ in the third quarter, and ⊥ in the bottom quarter.
> The third variable is assigned ⊤ in the top eight, ⊥ in the second eighth...
> ...
> The last variable is assigned alternating instances of ⊤ and ⊥.

So, in an 128 row truth table (7 variables), the first variable would get 64 ⊤s and 64 ⊥s, the second variable would get 32 ⊤s, 32 ⊥s, 32 ⊤s, and 32 ⊥s, the third variable would alternate ⊤s and ⊥s in groups of 16, the fourth variable would alternate ⊤s and ⊥s in groups of 8s... and the seventh variable would alternate single instances of ⊤s and ⊥s.
It does not matter which variables we take as first, second, third, etc., but it is conventional that we work from left to right.
Remember that every instance of the same variable letter gets the same assignment of truth values.

III. Exercises A. Construct truth tables for each of the following propositions.

1. ~P ⊃ Q
2. (P ≡ P) ⊃ P
3. ~Q ∨ (P ⊃ Q)

IV. Classifying propositions using truth tables

Compare the following propositions:

1. I exist.
2. I am here, now.
3. I am in New York.
4. I am in Canada.
5. 2+2=4
6. 2+2=5

1-3, and 5, are true; 4 and 6 are false.
Still, we can distinguish between necessary truths (1, 2, and 5), and merely contingent ones (3); and between necessary falsehoods (6) and merely contingent ones (4).
Furthermore, some compound propositions are necessarily true or false, independently of the truth values of their component propositions.
We can use truth tables to make distinctions among tautologies, contingencies, and contradictions.

Consider again the truth table for ‘P ⊃ P’

P	⊃	P
⊤	⊤	⊤
⊥	⊤	⊥

This is a *tautology*: statement that is always true.
We saw a couple of other tautologies in our last class.
The Law of the Excluded Middle says that any statement of the form ‘$\alpha \lor \sim\alpha$’ is a tautology.
The Law of Non-Contradiction says that any statement of the form ‘$\sim(\alpha \bullet \sim\alpha)$’ is a tautology.
Tautologies are the theorems of propositional logic.
They are sometimes call logical truths.
Here is a tautology in English:

> ‘Either the Phillies win the World Series this year, or they don’t.’

Not all necessary truths are tautologies, but all tautologies are necessary truths.
In particular, sentences 1, 2, and 5 are not logical truths, even if they are necessarily true.
For most, perhaps all, necessary truths, there are some philosophers who claim that they are really contingent.
But, the logical truths are among the least controversial.

Here is a longer tautology: ‘[(P ⊃ Q) · (Q ⊃ R)] ⊃ (P ⊃ R)’

[(P	⊃	Q)	·	(Q	⊃	R)]	⊃	(P	⊃	R)
⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤
⊤	⊤	⊤	⊥	⊤	⊥	⊥	⊤	⊤	⊥	⊥
⊤	⊥	⊥	⊥	⊥	⊤	⊤	⊤	⊤	⊤	⊤
⊤	⊥	⊥	⊥	⊥	⊤	⊥	⊤	⊤	⊥	⊥
⊥	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊥	⊤	⊤
⊥	⊤	⊤	⊥	⊤	⊥	⊥	⊤	⊥	⊤	⊥
⊥	⊤	⊥	⊤	⊥	⊤	⊤	⊤	⊥	⊤	⊤
⊥	⊤	⊥	⊤	⊥	⊤	⊥	⊤	⊥	⊤	⊥

Note that tautologies like this are not true in themselves.
Rather, they are true on all substitutions of propositions for the variables.

Only a small portion of the sentences of propositional logic are tautologies.
Consider the truth table for ‘P ∨ ~Q’

P	∨	~	Q
⊤	⊤	⊥	⊤
⊤	⊤	⊤	⊥
⊥	⊥	⊥	⊤
⊥	⊤	⊤	⊥

This is a *contingency*: statement that may or may not be true.
It is true in at least one row of the truth table; it is false in at least one row.
The truth of the complex proposition is contingent (depends) on the values of the component premises.
Most wffs will be contingent.

Consider the truth table for ‘P · ~P’

P	·	~	P
⊤	⊥	⊥	⊤
⊥	⊥	⊤	⊥

This is a *self-contradiction*: statement that is never true.

Here is another self-contradiction: ‘(~P ⊃ Q) ≡ ~(Q ∨ P)’:

(~	P	⊃	Q)	≡	~	(Q	∨	P)
⊥	⊤	⊤	⊤	⊥	⊥	⊤	⊤	⊤
⊥	⊤	⊤	⊥	⊥	⊥	⊥	⊤	⊤
⊤	⊥	⊤	⊤	⊥	⊥	⊤	⊤	⊥
⊤	⊥	⊥	⊥	⊥	⊤	⊥	⊥	⊥

V. Exercises B. Classify each proposition as tautologous, contingent, or self-contradictory.

1. ~A ⊃ ~A
2. B · (B ∨ F)
3. (~D · E) · (E ⊃ D)

VI. Classifying pairs of sentences using truth tables

Consider ‘(A ∨ B) ≡ (~B ⊃ A)’.

(A	∨	B)	≡	(~	B	⊃	A)
⊤	⊤	⊤	⊤	⊥	⊤	⊤	⊤
⊤	⊤	⊥	⊤	⊤	⊥	⊤	⊤
⊥	⊤	⊤	⊤	⊥	⊤	⊤	⊥
⊥	⊥	⊥	⊤	⊤	⊥	⊥	⊥

It is a tautology.
Eliminate the biconditional, and consider the two remaining halves as separate statements:

A	∨	B		~	B	⊃	A
⊤	⊤	⊤		⊥	⊤	⊤	⊤
⊤	⊤	⊥		⊤	⊥	⊤	⊤
⊥	⊤	⊤		⊥	⊤	⊤	⊥
⊥	⊥	⊥		⊤	⊥	⊥	⊥

These two statements are *logically equivalent*: Two or more statements with identical truth values in every row of the truth table.
The concept of logical equivalence will help us understand some left-over issues about translation, in our next class.

For now, consider some other relations among propositions.

Consider ‘A ∨ ~B’ and ‘B · ~A’.

A	∨	~	B		B	·	~	A
⊤	⊤	⊥	⊤		⊤	⊥	⊥	⊤
⊤	⊤	⊤	⊥		⊥	⊥	⊥	⊤
⊥	⊥	⊥	⊤		⊤	⊤	⊤	⊥
⊥	⊤	⊤	⊥		⊥	⊥	⊤	⊥

These statements form a *contradiction*: Two statements with opposite truth values in all rows of the truth table.
Note that the biconditional connecting the two statements of a contradiction is self-contradictory.
‘P · ~P’ is a simple contradiction, with common use.
In English: “It’s raining. It’s not raining.”
A person who makes both statements together has to be wrong about at least one of them.

Consider ‘E ⊃ D’ and ‘~E · D’.

E	⊃	D		~	E	·	D
⊤	⊤	⊤		⊥	⊤	⊥	⊤
⊤	⊥	⊥		⊥	⊤	⊥	⊥
⊥	⊤	⊤		⊤	⊥	⊤	⊤
⊥	⊥	⊥		⊤	⊥	⊥	⊥

These statements are neither contradictory (see rows 2, 3, and 4) nor logically equivalent (see row 1).
But a person who makes both statements can be making true statements. (See row 3).
It depends on what the substitutions are (for E and D).

If two statements are neither logically equivalent nor contradictory, they may be consistent or inconsistent.

> *Consistent*: Can be true together, for at least one valuation (one row of the table).
> *Inconsistent*: Not consistent. I.e. there is no row of the truth table in which both statements are true.

Here are an inconsistent pair: ‘E · F’ and ‘~(E ⊃ F)’

E	·	F		~	(E	⊃	F)
⊤	⊤	⊤		⊥	⊤	⊤	⊤
⊤	⊥	⊥		⊤	⊤	⊥	⊥
⊥	⊥	⊤		⊥	⊥	⊤	⊤
⊥	⊥	⊥		⊥	⊥	⊤	⊥

Note that the conjunction of two inconsistent statements is a self-contradiction.
When comparing two propositions, first look for the stronger conditions: logical equivalence and contradiction.
Then, if these fail, look for the weaker conditions: consistency and inconsistency.

The difference between two sentences which are inconsistent and two sentences which are contradictory is subtle.
In both cases, the pair of sentences can not be true together.
The difference is whether the pair can be false in the same conditions.
Contradictory pairs always have opposite truth values.
Inconsistent pairs may have truth conditions in which they are both false.
When we are making assertions, and aiming at the truth, it is generally just as bad to make inconsistent assertions as it is to make contradictory assertions.

VII. Exercises C. Are the statements logically equivalent or contradictory? If neither, are they consistent or inconsistent?

1.	A ⊃ ~B	~(B · A)
2.	A · ~B	B · ~A
3.	B · A	A ⊃ ~B
4.	A ≡ B	~(A ∨ B)
5.	A ∨ (B · D)	~A · ~(B ∨ ~D)

VIII. Solutions

Answers to Exercises A

1.

~	P	⊃	Q
⊥	⊤	⊤	⊤
⊥	⊤	⊤	⊥
⊤	⊥	⊤	⊤
⊤	⊥	⊥	⊥

2.

(P	≡	P)	⊃	P
⊤	⊤	⊤	⊤	⊤
⊥	⊥	⊥	⊥	⊥

3.

~	Q	∨	(P	⊃	Q)
⊥	⊤	⊤	⊤	⊤	⊤
⊥	⊤	⊤	⊥	⊤	⊤
⊤	⊥	⊤	⊤	⊥	⊥
⊤	⊥	⊤	⊥	⊤	⊥

Answers to Exercises B

1. Tautologous
2. Contingent
3. Contradictory

Answers to Exercises C

1. Logically equivalent
2. Inconsistent
3. Contradictory
4. Consistent
5. Inconsistent