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Translation Using Identity II (§8.7)

I. Special Properties of the Identity Predicate

We did not extended our language **F** by introducing the identity predicate.
We only set aside a particular two-place predicate, adding a new shorthand for it.
We do not need any new formation rules, though we should clarify how the shorthand works.
In particular, formulas like ‘a=b’ are really short for ‘Iab’.
Since we do not put brackets around ‘Iab’, we should not put brackets around ‘a=b’ either.
Hurley makes several errors about the syntax in the exercises section, for example at §8.7: II.1:

(x)(x=a)

This expression is not a wff, and should be written:

(x)x=a

While the identity predicate needs no new syntactical rules, we will introduce new derivation rules governing the predicate.
Technically, we are introducing a new deductive system which uses the same language **F**.

There is one new rule name we will use in derivations, ‘ID’.
‘ID’ refers to three different rules governing the identity predicate.
In Hurley’s deductive system, these rules apply only to constants, though I think no error will come from applying them also to variables.

ID Rule #1. Reflexivity: α=α

Any constant is, and stands for an object which is, identical to itself.
Thus, we can add, in any proof, a statement of this form.

ID Rule #2. Symmetry: α=β :: β=α

Identity is commutative.

ID Rule #3. Indiscernibility of Identicals

Consider again Superman and Clark Kent: s=c
We know that the two people are the same, so anything true of one, is true of the other.
This property is called Leibniz’s law, or the Law of the Indiscernibility of Identicals:

$$(x)(y)[(x=y) \supset (\mathscr{F}x \equiv \mathscr{F}y)]$$

Written as a rule of inference, we get:

$\mathscr{F}\alpha$
$\alpha=\beta \quad / \mathscr{F}\beta$

The third rule indicates that if $\alpha=\beta$, then, you may rewrite any formula containing α with β in the place of α throughout.
Be careful not to confuse the indiscernibility of identicals, which is pretty safe, with Leibniz's contentious claim of the identity of indiscernibles.
The latter relies on Leibniz's contentious assertion of the Principle of Sufficient Reason.

II. 'Exactly'

The line between logic and mathematics is thin.
Two early developers of modern logic, Frege and Russell, believed mathematics to be just logic written in a more complicated form.
Frege and Russell believed that written in the proper formal language, all statements of mathematics could be reduced to logical truths.
The claim that mathematics is just a more complex form of logic has become known as logicism.
Logicism was widely proclaimed to be a failure, since mathematics requires non-logical axioms.
Normally, we extend logical systems to mathematical ones by including one more element to the language, $\in$, standing for set inclusion, and axioms governing set theory.
Mathematics is uncontroversially reducible to logic plus set theory, in at least a formal sense.
There are contemporary philosophers who continue to work on the original Fregean logicist project; they are known as neo-logicists, or neo-Fregeans.

Despite the failure of logicism as it was conceived by Frege and Russell, we can gerrymander some mathematical concepts with logic proper.
In our last translation class, we translated sentences including the phrases 'at least' and 'at most'.
We can construct adjectival uses of the natural numbers by using identity.
To say that there are exactly n of some object, we combine at-least and at-most clauses.
That is, if we want to say that there are exactly two chipmunks in the yard, we just say that there are both at least two chipmunks and at most two chipmunks.

1. There are exactly two chipmunks in the yard.
   $(\exists x)(\exists y)\{Cx \bullet Yx \bullet Cy \bullet Yy \bullet x\neq y \bullet (z)[(Cz \bullet Yz) \supset (z=x \lor z=y)]\}$

2. There are at least two applicants.
   $(\exists x)(\exists y)[(Ax \bullet Ay) \bullet x\neq y]$

3. There are at most two applicants.
   $(x)(y)(z)[(Ax \bullet Ay \bullet Az) \supset (x=y \lor x=z \lor y=z)]$

4. There are exactly two applicants.
   $(\exists x)(\exists y)\{[Ax \bullet Ay \bullet x\neq y] \bullet (z)[Az \supset (z=x \lor z=y)]\}$

5. Two is the only even prime number.
   $Et \bullet Pt \bullet Nt \bullet (x)[(Ex \bullet Px \bullet Nx) \supset x=t]$

6. There is exactly one even prime number.
   $(\exists x)\{(Ex \bullet Px \bullet Nx) \bullet (y)[(Ey \bullet Py \bullet Ny) \supset y=x]\}$

7. There are exactly three aardvarks on the log.

$$(\exists x)(\exists y)(\exists z)\{Ax \bullet Lx \bullet Ay \bullet Ly \bullet Az \bullet Lz \bullet x{\neq}y \bullet \bullet\ x{\neq}z \bullet y{\neq}z \bullet (w)[(Aw \bullet Lw) \supset (w{=}x \vee w{=}y \vee w{=}z]\}$$

You may notice that these mock-up numerical sentences get very long very quickly.
To abbreviate, logicians sometimes introduce special shorthand quantifiers.

$(\exists 1x)$, $(\exists 2x)$, $(\exists 3x)$...

Sometimes the above quantifiers are taken to indicate that there are at least the number indicated.
To indicate exactly a number, '!' is used.
For exactly one thing, people sometimes write '$(\exists !x)$'.
For more things, we can insert the number and the '!'.

$(\exists 1!x)$, $(\exists 2!x)$, $(\exists 3!x)$...

These abbreviations are useful for translation (though not on your exams!).
Once we want to make inferences using the numbers, we have to unpack their longer form.

III. Bertrand Russell's analysis for definite descriptions:

One important use of the identity predicate is in a solution to a philosophical problem.
Consider:

8. The king of America is bald.

We might translate it as 'Bk'.
'Bk' is false, since there is no king of America.
So, '~Bk' should be true, since it's the negation of a false statement.
But '~Bk' seems to be a perfectly reasonable regimentation of:

9. The king of America is not bald.

9 has the same grammatical form as 10.

10. Devendra Banhart is not bald.

10 entails that Devendra Banhart has hair.
So, 9 may reasonably be taken to imply that the king of America has hair.
In fact, we want both 8 and 9 to be false.
But, the conjunction of their negations is a contradiction:

11. $\sim Bk \bullet \sim\sim Bk$

We had better regiment 8 and 9 differently.
'The king of America' is a definite description.
It refers to one specific object without using a name.

There are two ways to refer to an object.
We can use the name of the object, or we can describe it (e.g. the person who, the thing that)

Both 8 and 9 use definite descriptions to refer to an object.
They are false, due to a false presupposition in the description.
Descriptions may be complex, and we can unpack them.
8 entails three simpler expressions:

A. There is a king of America.	$(\exists x)Kx$
B. There is only one king of America.	$(y)(Ky \supset y=x)$
C. That thing is bald.	Bx

Putting it all together, so that every term is within the scope of the existential quantifier, we get:

12. $(\exists x)[Kx \bullet (y)(Ky \supset y=x) \bullet Bx]$

So, 8 is false because clause A is false.
9 is also false, for the same reason.

13. $(\exists x)[Kx \bullet (y)(Ky \supset y=x) \bullet \sim Bx]$

The negation only affects the third clause.
The first clause is the same in 12 and 13, and still false.
Further, when we conjoin 12 and 13, we do not get a contradiction, as we did in 11.

14. $(\exists x)[Kx \bullet (y)(Ky \supset y=x) \bullet Bx] \bullet (\exists x)[Kx \bullet (y)(Ky \supset y=x) \bullet \sim Bx]$

14 is no more problematic than.

$(\exists x)Px \bullet (\exists x)\sim Px$ e.g. Some things are purple, and some things are not purple.

One might worry that 12 and 13 are still problematic, since the uniqueness clauses in 12 and 13 seem to make it the case that we are talking about the same thing having both the properties of baldness and lacking that property.
Let's see why this is not so.
First, note that we are only prepared to assert the negations of 12 and 13:

12'. $\sim(\exists x)[Kx \bullet (y)(Ky \supset y=x) \bullet Bx]$
13'. $\sim(\exists x)[Kx \bullet (y)(Ky \supset y=x) \bullet \sim Bx]$

If we were to assert both 12 and 13, instead of their negations, we would be able to derive a contradiction.
But, the contradiction would be present in both 8 and 9, too.
It is no error in a logic if it derives a contradiction from contradictory statements!
The problem arises only because we want to assert the negations of 8 and 9, and the simple regimentation leads to the contradiction at 11.
Now, let's unpack 12' and 13', and see if we can get to a contradiction.
Working from 12', we can get:

15. $(x){\sim}[Kx \bullet (y)(Ky \supset y{=}x) \bullet Bx]$	12', CQ
16. $(x)[{\sim}Kx \lor {\sim}(y)(Ky \supset y{=}x) \lor {\sim}Bx]$	15, DM
17. $(x)[{\sim}Kx \lor (\exists y){\sim}(Ky \supset y{=}x) \lor {\sim}Bx]$	16, CQ
18. $(x)[{\sim}Kx \lor (\exists y){\sim}({\sim}Ky \lor y{=}x) \lor {\sim}Bx]$	17, Impl
19. $(x)[{\sim}Kx \lor (\exists y)(Ky \bullet {\sim}y{=}x) \lor {\sim}Bx]$	18, DM, DN

Doing the same from 13', we get:

20. $(x){\sim}[Kx \bullet (y)(Ky \supset y{=}x) \bullet {\sim}Bx]$	13', CQ
21. $(x)[{\sim}Kx \lor {\sim}(y)(Ky \supset y{=}x) \lor Bx]$	20, DM, DN
22. $(x)[{\sim}Kx \lor (\exists y){\sim}(Ky \supset y{=}x) \lor Bx]$	21, CQ
23. $(x)[{\sim}Kx \lor (\exists y){\sim}({\sim}Ky \lor y{=}x) \lor Bx]$	22, Impl
24. $(x)[{\sim}Kx \lor (\exists y)(Ky \bullet {\sim}y{=}x) \lor Bx]$	23, DM, DN

The conjunction of 19 and 22 will not lead to contradiction, even if we instantiate both to the same constant and combine them.

25. ${\sim}Ka \lor (\exists y)(Ky \bullet {\sim}y{=}a) \lor {\sim}Ba$	19, UI
26. ${\sim}Ka \lor (\exists y)(Ky \bullet {\sim}y{=}a) \lor Ba$	24, UI
27. $\{{\sim}Ka \lor (\exists y)(Ky \bullet {\sim}y{=}a) \lor {\sim}Ba\} \bullet \{{\sim}Ka \lor (\exists y)(Ky \bullet {\sim}y{=}a) \lor {\sim}Ba\}$	25, 26, Conj
28. ${\sim}Ka \lor (\exists y)(Ky \bullet {\sim}y{=}a) \lor (Ba \bullet {\sim}Ba)$	27, Dist

Thus, by asserting the negations of 12 and 13, we are asserting only either that there is no king of America, or that there is more than one king of America, or that some thing is both bald and not bald.

Here is another example using definite descriptions:

29. The country called a sub-continent is India.

We can again divide it into three clauses:

A. There is a country called a sub-continent.
B. There is only one such country.
C. That country is identical with India.

So, we regiment it as:

30. $(\exists x)\{(Cx \bullet Sx) \bullet (y)[(Cy \bullet Sy) \supset y{=}x] \bullet x{=}i\}$

Here is Russell's original example:

31. The author of Waverly was a genius: $(\exists x)\{Wx \bullet (y)[Wy \supset y{=}x] \bullet Gx\}$

V. More translation examples, including some review

1. Everything is identical with itself.
   $(x)x=x$

2. Nothing is distinct from itself.
   $(x)\sim\sim x=x$

3. Everything is identical with something.
   $(x)(\exists y)x=y$

4. John loves Mary.
   Ljm

5. John only loves Mary.
   $Ljm \cdot (x)(Ljx \supset x=m)$

6. Only John loves Mary.
   $Ljm \cdot (x)(Lxm \supset x=j)$

7. Everyone loves Mary.
   $(x)(Px \supset Lxm)$

8. Everyone except John loves Mary.
   $\sim Ljm \cdot (x)[(Px \cdot x \neq j) \supset Lxm]$

9. Everyone deems all Beatles' records except *Let It Be* to be classics.
   $\sim(\exists x)(Px \bullet Dxl) \bullet (x)\{Px \supset (y)[(By \bullet Ry \bullet y \neq l) \supset Dxy]\}$

10. Adriana is a bigger mouse than Rene. (Bxy: x is bigger than y)
    $Ma \bullet Mr \bullet Bar$

11. Adriana is the biggest mouse.
    $Ma \bullet (x)[(Mx \bullet \sim x=a) \supset Bax]$

12. Bill Gates is the geek with the most money.
    $Gg \cdot (x)[(Gx \cdot x \neq g) \supset Mgx]$

V. Exercises. Translate into first-order logic, using the identity predicate where required.

1. All prime numbers are odd except the number two.
2. There is at least one mouse bigger than Rene.
3. There are at least two mice bigger than Rene.
4. There are at least three mice bigger than Rene.
5. Rene is the smallest mouse.
6. Syracuse is the nearest major city.
7. There are at least two odd prime numbers.
8. At most two persons invented the airplane.
9. There is exactly one dollar bill in my wallet. (Dx, Wx)
10. There are at least four students in the course. (Sx, Cx)
11. There are exactly three applicants.
12. The murderer was Colonel Mustard. (m, Mx)

VI. Solutions

1. $(x)[(Px \bullet Nx \bullet {\sim}x{=}t) \supset Ox]$
2. $(\exists x)(Mx \bullet Bxr)$
3. $(\exists x)(\exists y)(Mx \bullet My \bullet Bxr \bullet Byr \bullet x{\neq}y)$
4. $(\exists x)(\exists y)(\exists z)(Mx \bullet My \bullet Mz \bullet Bxr \bullet Byr \bullet Bzr \bullet x{\neq}y \bullet x{\neq}z \bullet y{\neq}z)$
5. $(x)[(Mx \bullet {\sim}x{=}r) \supset Bxr]$
6. $(x)[(Mx \bullet x{\neq}s) \supset Nsx]$
7. $(\exists x)(\exists y)(Ox \bullet Px \bullet Nx \bullet Oy \bullet Py \bullet Ny \bullet {\sim}x{=}y)$
8. $(x)(y)(z)[(Px \bullet Ix \bullet Py \bullet Iy \bullet Pz \bullet Iz) \supset (x{=}y \vee x{=}z \vee y{=}z)]$
9. $(\exists x)\{(Dx \bullet Wx) \bullet (y)[(Dy \bullet Wy) \supset y{=}x]\}$
10. $(\exists x)(\exists y)(\exists z)(\exists w)(Sx \bullet Cx \bullet Sy \bullet Cy \bullet Sz \bullet Cz \bullet Sw \bullet Cw \bullet x{\neq}y \bullet x{\neq}z \bullet x{\neq}w \bullet y{\neq}z \bullet y{\neq}w \bullet z{\neq}w)$
11. $(\exists x)(\exists y)(\exists z)\{[Ax \bullet Ay \bullet Az \bullet x{\neq}y \bullet x{\neq}z \bullet y{\neq}z] \bullet (w)[Aw \supset (w{=}x \vee w{=}y \vee w{=}z)]\}$
12. $(\exists x)[Mx \bullet (y)(My \supset y{=}x) \bullet x{=}m]$