Philosophy 240: Symbolic Logic
Fall 2010
Mondays, Wednesdays, Fridays: 9am - 9:50am

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Invalidity in Predicate Logic (§8.5)

I. Invalidity In PL

We have studied a proof-theoretic method for showing that an argument in **M** is valid.
We have not studied a method for showing that an argument in **M** is invalid.

Recall how we proved invalidity in propositional logic.
Consider an argument:

1. A ⊃ B
2. ~(B · A) /A ≡ B

We lined up the premises and conclusion:

A ⊃ B / ~(B · A) // A ≡ B

We then assigned truth values to the component sentences to form a counterexample.
A counterexample is a valuation which makes the premises true and the conclusion false.

A	⊃	B	/	~	(B	·	A)	//	A	≡	B
⊥	⊤	⊤		⊤	⊤	⊥	⊥		⊥	⊥	⊤

So, the argument is shown invalid when A is false and B is true.
We will adapt this method for first-order logic.

II. The Method of Finite Universes

If an argument is valid, then it is valid, no matter what we choose as our domain of interpretation.
Logical truths are true in all possible universes.
Even if our domain has only one member, or two or three, valid arguments must be valid.

Consider the following invalid argument:

(x)(Wx ⊃ Hx)
(x)(Ex ⊃ Hx) / (x)(Wx ⊃ Ex)

We will start by choosing a domain of one object in the universe.
We will call it ‘a’.

Then:

(x)(Wx ⊃ Hx)	is equivalent to	Wa ⊃ Ha
(x)(Ex ⊃ Hx)	is equivalent to	Ea ⊃ Ha
∴(x)(Wx ⊃ Ex)	is equivalent to	Wa ⊃ Ea

Now, assign truth values, as in the propositional case to make the premises true and the conclusion false:

Wa	⊃	Ha	/	Ea	⊃	Ha	//	Wa	⊃	Ea
⊤	⊤	⊤		⊥	⊤	⊤		⊤	⊥	⊥

So, the argument is shown invalid in a one-member universe, where Wa is true, Ha is true, and Ea is false.

A specification of the assignments of truth values to the atomic sentences of the theory, as in the previous sentence, is called a counter-example.
Be careful not to confuse this use of ‘counter-example’ with the counter-example method.
When I ask you to specify counter-examples in the method of finite universes, I am asking for assignments of truth values of the atomic sentences, given your chosen domain.

The method of finite universes works for more complex arguments, as well.

1. (x)[Ux ⊃ (Tx ⊃ Wx)]
2. (x)[Tx ⊃ (Ux ⊃ ~Wx)]
3. (∃x)(Ux · Wx)

∴(∃x)(Ux · Tx)

Ua	⊃	(Ta	⊃	Wa)	/	Ta	⊃	(Ua	⊃	~	Wa)	/	Ua	•	Wa	//	Ua	•	Ta
⊤	⊤	⊥	⊤	⊤		⊥	⊤	⊤	⊥	⊥	⊤		⊤	⊤	⊤		⊤	⊥	⊥

Counter-example: The argument is shown invalid in a one-member universe, where Ua is true; Ta is false; and Wa is true.

Not all invalid arguments are shown invalid in a one-member universe.
All we need is one universe in which they are shown invalid, to show that they are invalid.
Even if an argument has no counter-example in a one-member universe, it might still be invalid!

III. Universes of More Than One Member

Consider the following invalid argument:

(x)(Wx ⊃ Hx)
(∃x)(Ex · Hx) / (x)(Wx ⊃ Ex)

In a one-object universe, we have:

Wa	⊃	Ha	/	Ea	·	Ha	//	Wa	⊃	Ea
				⊥				⊤	⊥	⊥

There is no way to construct a counterexample in a one-member domain.
But the argument is invalid.
We have to consider a larger universe.

If there are two objects in a universe, a and b:

(x)$\mathscr{F}$x	becomes	$\mathscr{F}$a · $\mathscr{F}$b	because every object has $\mathscr{F}$
(∃x)$\mathscr{F}$x	becomes	$\mathscr{F}$a ∨ $\mathscr{F}$b	because only some objects have $\mathscr{F}$

If there are three objects in a universe, then

(x)$\mathscr{F}$x	becomes	$\mathscr{F}$a · $\mathscr{F}$b · $\mathscr{F}$c
(∃x)$\mathscr{F}$x	becomes	$\mathscr{F}$a ∨ $\mathscr{F}$b ∨ $\mathscr{F}$c

Returning to the problem...
In a universe of two members, we represent the argument is equivalent to:

(Wa ⊃ Ha) · (Wb ⊃ Hb) / (Ea · Ha) ∨ (Eb · Hb) // (Wa ⊃ Ea) · (Wb ⊃ Eb)

Now, assign values to each of the terms to construct a counterexample.

(Wa	⊃	Ha)	·	(Wb	⊃	Hb)	/	(Ea	·	Ha)	∨	(Eb	·	Hb)
⊤	⊤	⊤	⊤	⊥	⊤	⊤		⊥	⊥	⊤	⊤	⊤	⊤	⊤

//	(Wa	⊃	Ea)	·	(Wb	⊃	Eb)
	⊤	⊥	⊥	⊥	⊥	⊤	⊤

The argument is shown invalid in a two-member universe, when

Wa: true	Wb: false
Ha: true	Hb: true
Ea: false	Eb: true

IV. Constants

When expanding formulas into finite universes, constants get rendered as themselves.
That is, we don't expand a term with a constant when moving to a larger universe.
Consider:

(∃x)(Ax · Bx)
Ac /Bc

We can't show it invalid in a one-member universe.

Ac	·	Bc	/	Ac	//	Bc
	⊥	⊥				⊥

We can generate a counter-example in a two-member universe.

(Ac	·	Bc)	∨	(Aa	·	Ba)	/	**Ac**	//	**Bc**
⊤	⊥	⊥	⊤	⊤	⊤	⊤		⊤		⊥

This argument is shown invalid in a two-member universe, when

Ac: true Bc: false
Aa: true Ba: true

Some arguments need three, four, or even infinite models to be shown invalid.

V. Propositions Whose Main Connective is Not a Quantifier

Consider the following argument:

(∃x)(Px · Qx)
(x)Px ⊃ (∃x)Rx
(x)(Rx ⊃ Qx) /(x)Qx

In a one-member universe, this argument gets rendered as:

Pa · Qa / Pa ⊃ Ra / Ra ⊃ Qa // Qa

But there is no counter-example in a one-member universe.

Pa	·	Qa	/	Pa	⊃	Ra	/	Ra	⊃	Qa	//	**Qa**
	⊥	⊥										⊥

In a two-member universe, note what happens to the second premise:

(Pa · Qa) ∨ (Pb · Qb) / (Pa · Pb) ⊃ (Ra ∨ Rb) / (Ra ⊃ Qa) · (Rb ⊃ Qb) // Qa · Qb

Each quantifier is unpacked independently.
The main connective, the conditional, remains the main connective.
We can clearly see here the difference between instantiation and translation into a finite universe.
We can construct a counterexample for this argument in a two-member universe:

(Pa	·	Qa)	∨	(Pb	·	Qb)	/	(Pa	·	Pb)	⊃	(Ra	∨	Rb)
	⊥	⊥	⊤	⊤	⊤	⊤				⊤	⊤	⊥	⊤	⊤

/	(Ra	⊃	Qa)	·	(Rb	⊃	Qb)	//	Qa	·	Qb
	⊥	⊤	⊥	⊤	⊤	⊤	⊤		⊥	⊥	⊤

This argument is shown invalid in a two-member universe, when
Pa: either true or false Pb: true
Qa: false Qb: true
Ra: false Rb: true

(There is another solution. Can you construct it?)

VI. Exercises A. Show each of the following arguments invalid by generating a counter-example using the method of finite universes.

1. 1. (x)(Ex ⊃ Fx)
   2. (∃x)(Gx · ~Fx) / (∃x)(Ex · ~Gx)

2. 1. (x)(Bx ⊃ ~Dx)
   2. ~Bj / Dj

3. 1. (x)(Hx ⊃ ~Ix)
   2. (∃x)(Jx · ~Ix) / (x)(Hx ⊃ Jx)

4. 1. (x)(Kx ⊃ ~Lx)
   2. (∃x)(Mx · Lx) / (x)(Kx ⊃ ~Mx)

5. 1. (∃x)(Px · Qx)
   2. (x)(Qx ⊃ ~Rx)
   3. Pa / (x)~Rx

6. 1. (x)(Ax ⊃ Bx)
   2. (∃x)(Dx · Bx)
   3. (∃x)(Dx · ~Bx) / (x)(Ax ⊃ Dx)

VII. The Informal Counter-Example Method

Hurley also presents an informal method to prove an argument invalid.
This method will not be on the test.
Consider:

(x)(Wx ⊃ Hx)
(x)(Ex ⊃ Hx) / (x)(Wx ⊃ Ex)

We can provide an interpretation of the predicates that yields true premises but a false conclusion.

Wx: x is a whale
Ex: x is an elephant
Hx: x is heavy

So, 'all whales are heavy' and 'all elephants are heavy' are both true.
But, 'all whales are elephants' is false.

The informal counter-example method is fine for shorter, simpler arguments.
Some of you are probably smart enough to come up with something for an argument like:

1. (x)[Ux ⊃ (Tx ⊃ Wx)]
2. (x)[Tx ⊃ (Ux ⊃ ~Wx)]
3. (∃x)(Ux · Wx)

∴(∃x)(Ux · Tx)

But, the finite universes method requires less ingenuity.
We will not use the informal counter-example method.
And, when I ask for a counter-example, I do not mean for you to devise an interpretation like these.
I want an assignment of truth values to the component premises in a chosen finite domain.

VIII. Exercises B. Show invalid, using the informal counterexample method:

1.
 1. (x)(Ax ⊃ Bx)
 2. Bj / Aj

2.
 1. (∃x)(Ax · Bx)
 2. Aa / Ba

3.
 1. (x)(Hx ⊃ Ix)
 2. (x)(Hx ⊃ ~Jx) / (x)(Ix ⊃ ~Jx)

IX. Solutions

Sample answers to Exercises A
1. Shown invalid in a one-member universe, where Ga: true; Ea: false; Fa: false
2. Shown invalid in a one-member universe, where Bj: false; Dj: false
3. Shown invalid in a two-member universe, where Ha: true; Ia: false; Ja: false; Hb: true or false; Ib: false; Jb: true
4. Shown invalid in a two-member universe, where Ka: false; La: true; Ma: true; Kb: true; Lb: false; Mb: true.
5. Shown invalid in a two-member universe, where Pa: true; Qa: false; Ra: true; Pb: true; Qb: true; Rb: false
6. Shown invalid in a three-member universe, where Aa: true; Ba: true; Da: false; Ab: true or false; Bb: true; Db: true; Ac: false; Bc: false; Dc: true

Sample answers to Exercises B
1. Ax: x is an apple; Bx: x is a fruit; j: a pear
2. Ax: x is a Met; Bx: x is a pitcher; a: Carlos Delgado
3. Hx: x is a desk; Ix: x has legs; Jx: x has arms