Hamilton College
Russell Marcus
rmarcus1@hamilton.edu

Class 3 - September 1
Truth Functions (§6.2)

## I. Introduction

When constructing a formal system, we first provide formation rules, or a syntax, for that system.
Then, we may either use the system by introducing transformation or inference rules, or we can interpret the system.
We will study inferences later.
For now, we will discuss the interpretations, or semantics, of our system.
In standard propositional logic, we interpret propositional variables by assigning truth values to them.
We'll start our study of the semantics of propositional logic by looking at how we calculate the truth-value of a complex proposition on the basis of the truth values of its component sentences.
Consider:

> Either *The Hurt Locker* or *Avatar* won the Oscar for Best Picture, but *The Hurt Locker* won if, and only if, George Clooney did not win the Oscar for Best Actor.

Translated into Propositional Logic: (H ∨ A) · (H ≡ ~C)
We know the values of the component propositions H, A, and C

> H is true
> A is false
> C is false

But what is the value of the complex proposition?
Definition: the *truth value of a complex proposition* is the truth value of its main connective.

## II. Basic truth tables

We can derive the truth value of a complex proposition given the truth values of its component propositions using the basic truth tables for each connective.
This fact is called truth-functional compositionality.

**Negation**
Note that while '2+2=4' is true, its negation, '2+2≠4' is false.
Also, while '2+2=5' is false, its negation, ' 2+2≠5' is true.
We summarize these results using a truth table.

| ~ | α |
|---|---|
| ⊥ | ⊤ |
| ⊤ | ⊥ |

There are two kinds of symbols in this truth table that are new to us.
I use letters of the Greek alphabet to stand for any propositional variable.
The Greek letters are metalogical variables.
They are symbols of a language in which we talk about our logic.
Similarly, I use metalogical variables '⊤' to stand for 'true' and '⊥' to stand for 'false'.
Truth and falsity are the two semantic values we will assign to variables.
There are other systems of logic which use three or more semantic values.
We'll call them truth values, but we need not think of them that way.
'⊤' may be read 'top' and '⊥' may be read 'bottom', if we want to be very abstract about our semantics.

Hurley's truth tables look a little bit different.
For negation, Hurley's truth table looks as follows:

| ~ | *p* |
|---|---|
| F | T |
| T | F |

Hurley is using italicized lower-case English letters as metalogical variables which stand for propositional variables.
The advantage of using English, or Roman, letters, is that they are less scary than Greek letters.
The disadvantage is that we will be using lower-case letters in our system of predicate logic.
I prefer not to confuse matters by using the same characters for two different purposes.
Similarly, Hurley uses 'T' and 'F' instead of '⊤' and '⊥'.
But, since 'T' and 'F' are propositional variables, using '⊤' and '⊥' keeps things straight.

Let's return to our truth table for negation.

| ~ | α |
|---|---|
| ⊥ | ⊤ |
| ⊤ | ⊥ |

The column under the 'α' represents all possible assignments of truth values to a single proposition.
The column under the '~' represents the values of the negation of that proposition in each row.
A truth table for a complex proposition containing one variable has two lines, since there are only two possible assignments of truth values.

**Conjunction**
Consider: ‘He likes logic and metaphysics.’
This statement is true if ‘He likes logic’ is true and ‘He likes metaphysics’ is true.
It is false otherwise.

| α | · | β |
|---|---|---|
| ⊤ | ⊤ | ⊤ |
| ⊤ | ⊥ | ⊥ |
| ⊥ | ⊥ | ⊤ |
| ⊥ | ⊥ | ⊥ |

Note that we need 4 lines to explore all the possibilities:
When both are true (row 1),
When one is true and the other is false (rows 2 and 3), and
When both are false (row 4).
With 3 variables, we need 8 lines, and with 4 variables, we need 16 lines.
How many rows would one need for 5 variables?
For n variables?

**Disjunction**
Consider: ‘She can get an A in either history or physics.’
We use an inclusive disjunction, on which this statement is false only when both component statements are false.

| α | ∨ | β |
|---|---|---|
| ⊤ | ⊤ | ⊤ |
| ⊤ | ⊤ | ⊥ |
| ⊥ | ⊤ | ⊤ |
| ⊥ | ⊥ | ⊥ |

**Material Implication**
Consider: ‘If you paint my house, then I will give you $500.'
When will this statement will be falsified?
It’s true if both the antecedent and consequent are true.
It’s false if the antecedent is true and the consequent is false.
If the antecedent is false, we consider this statement as unfalsified, and, thus, true.

| α | ⊃ | β |
|---|---|---|
| ⊤ | ⊤ | ⊤ |
| ⊤ | ⊥ | ⊥ |
| ⊥ | ⊤ | ⊤ |
| ⊥ | ⊤ | ⊥ |

**The Material Biconditional**
Consider: ‘Supplies rise if and only if demand falls.’
This is true if the component statements share the same truth value.
It is false if the components have different values.

| α | ≡ | β |
|---|---|---|
| ⊤ | ⊤ | ⊤ |
| ⊤ | ⊥ | ⊥ |
| ⊥ | ⊥ | ⊤ |
| ⊥ | ⊤ | ⊥ |

**III. Determining the truth value of a complex proposition**

The basic truth tables can be used to evaluate the truth value of any proposition built using the formation rules.

1. Assign truth values to each simple term.
2. Evaluate any negations of those terms.
3. Evaluate any connectives for which both values are known.
4. Repeat steps 2 and 3, working inside out, until you reach the main operator.

So, consider:
(A ∨ X) · ~B, given that A and B are true and X is false

First, assign the values to A, B, and X:

| (A | ∨ | X) | · | ~ | B |
|---|---|---|---|---|---|
| ⊤ | | ⊥ | | | ⊤ |

Next, evaluate the negation of B:

| (A | ∨ | X) | · | ~ | B |
|---|---|---|---|---|---|
| ⊤ | | ⊥ | | ⊤ | ⊤ |

Since you know the values of the disjuncts, you can next evaluate the disjunction:

| (A | ∨ | X) | · | ~ | B |
|---|---|---|---|---|---|
| ⊤ | ⊤ | ⊥ | | ⊥ | ⊤ |

Finally, you can evaluate the main connective, the conjunction:

| (A | ∨ | X) | · | ~ | B |
|---|---|---|---|---|---|
| ⊤ | ⊤ | ⊥ | ⊥ | ⊥ | ⊤ |

So, the proposition is false.

Returning to the problem from the beginning of the lesson: (H ∨ A) · (A ≡ ~C)

| (H | ∨ | A) | · | (H | ≡ | ~ | C) |
|---|---|---|---|---|---|---|---|
| ⊤ | ⊤ | ⊥ | ⊤ | ⊤ | ⊤ | ⊤ | ⊥ |

The proposition is false.

Consider these further examples:

1. A ⊃ (~X · ~Y), given that A is true and X and Y are false

| A | ⊃ | (~ | X | · | ~ | Y) |
|---|---|---|---|---|---|---|
| ⊤ | ⊤ | ⊤ | ⊥ | ⊤ | ⊤ | ⊥ |

The proposition is true

2. [(A · B) ⊃ Y] ⊃ [A ⊃ (C ⊃ Z)], given that A, B, and C are true, and Y and Z are false.

| [(A | · | B) | ⊃ | Y] | ⊃ | [A | ⊃ | (C | ⊃ | Z)] |
|---|---|---|---|---|---|---|---|---|---|---|
| ⊤ | ⊤ | ⊤ | ⊥ | ⊥ | ⊤ | ⊤ | ⊥ | ⊤ | ⊥ | ⊥ |

The proposition is true.

**IV. Exercises A**. Assume A, B, C are true and X, Y, Z are false. Evaluate the truth values of each:

1. Z ⊃ ~B
2. (B ≡ C) ⊃ ~A
3. B ⊃ (A ∨ C)
4. X ∨ (A · Y)
5. A ∨ ~A
6. Y ∨ ~Y
7. A · ~A
8. (A ⊃ Z) ∨ (~X ⊃ B)
9. [X · (A ∨ C)] ∨ ~[(X ∨ A) · (X ∨ C)]

**V. Exercises B**. Translate to propositional logic, and use your knowledge of the truth values of the component sentences to determine the truth values of the given complex propositions.

1. Mark Twain wrote *Huckleberry Finn* and Shakespeare wrote *Moby Dick*.
2. If Dickens was not American, then Proust was German.
3. It's not the case that Hemingway wrote both *The Old man and the Sea* and *The Great Gatsby*.
4. Steinbeck wrote *Of Mice and Men* if and only if Robert Frost didn't write 'The Wasteland'.
5. The assertion that neither Dostoevsky wrote both *Crime and Punishment* and *The Brothers Karamozov* nor Tolstoy wrote *War and Peace* is false.

**VI. Determining the truth values of complex propositions, when one component is unknown**

We have seen how to calculate the truth value of a complex, or compound, proposition when the truth values of the components are known.
This property of classical logic, that truth values of long expressions are always computable from the truth values of simpler expressions, is called truth-functional compositionality.

Sometimes you don't know truth values of one or more component variable.
(Soon we will dispense with the pretense that we know truth values of any un-interpreted letters.)
For purposes of this lesson, suppose that A, B, C are true; X, Y, Z are false; and P and Q are unknown.

Consider: P · A
If P is true, then we have:
⊤ · ⊤
which is true.
If P is false, then we have
⊥ · ⊤
which is false.
Since the truth value of the compound expression depends on the truth value of P, it too is unknown.

But consider: P ∨ A
If P is true, then we have
⊤ ∨ ⊤
which is true.
If P is false, then we have
⊥ ∨ ⊤
which is also true.
Since the truth value of the complex proposition is true in both cases, the value of that statement is true.

Similarly, consider: Q · Y
If Q is true, then we have
⊤ · ⊥
which is false.
If Q is false, then we have
⊥ · ⊥
which is also false.
Since the truth value of the complex proposition is false in both cases, the value of that statement is false.
If the truth values come out the same in each case, then the statement has that truth value.
If the values come out differently in different cases, then the truth value of the statement is unknown.

**VII. Exercises C**. Evaluate the truth value of each complex expression, using the same truth values as above.

1. ~(P · X) ⊃ Y
2. P ⊃ A
3. A ⊃ P
4. Q ∨ ~Z
5. P · ~P
6. Q ∨ ~Q
7. ~P ∨ (~X ∨ P)
8. [(P ⊃ X) ⊃ P] ⊃ P
9. (X ⊃ Q) ⊃ X

**VIII. Determining the truth values of complex propositions, when more than one component is unknown**

Lastly, one can have more than one unknown in a statement. If there are two unknowns, we must consider four cases.

Consider: $\sim(P \cdot Q) \vee P$
If P and Q are both true
$\sim(\top \cdot \top) \vee \top$
which is true.
If P is true and Q is false
$\sim(\top \cdot \bot) \vee \top$
which is true.
If P is false and Q is true
$\sim(\bot \cdot \top) \vee \bot$
which is true.
If P and Q are both false
$\sim(\bot \cdot \bot) \vee \bot$
which is again true.

Since all possible substitutions of truth values yield a true statement, the statement is true.

**IX. Exercises D**. Evaluate the truth value of each complex expression, using the same truth values as above.

1. $(P \cdot Q) \vee (\sim Q \vee \sim P)$
2. $(P \vee Q) \cdot (\sim B \vee Y)$
3. $(P \supset Q) \supset \{[P \supset (Q \supset A)] \supset (P \supset A)\}$

**X. Solutions**

Answers to Exercises A:

1. True
2. False
3. True
4. False
5. True
6. True
7. False
8. True
9. False

Answers to Exercises B:

1. False
2. False
3. True
4. True
5. True

Answers to Exercises C:

1. False
2. True
3. Unknown
4. True
5. False
6. True
7. True
8. True
9. False

Answers to Exercises D:

1. True
2. False
3. True