Philosophy 240: Symbolic Logic

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Semantics for Predicate Logic

Theories

- A *theory* is a set of sentences.
- A formal theory is a set of sentences of a formal language.
- We identify a theory by its theorems, the set of sentences provable within that theory.
- Many interesting formal theories are infinite.
  - Rules of inference generate an infinite number of theorems.

Constructing Formal Theories

1. Specify a language.
2. Add formation rules for wffs.

Syntax for **PL** and **M**

Vocabulary for **PL**

Capital letters A...Z
Five connectives: ~, •, ∨, ⊃ ≡
Punctuation: ( ), [ ], { }

Vocabulary for **M**

Capital letters A...Z used as one-place predicates
Lower case letters
  a, b, c,...u are used as constants.
  v, w, x, y, z are used as variables.
Five connectives: ~, •, ∨, ⊃ ≡
Quantifier: ∃
Punctuation: ( ), [ ], { }

Formation rules for Wffs of **PL**

1. A single capital English letter is a wff.
2. If α is a wff, so is ~α.
3. If α and β are wffs, then so are:
   (α · β)
   (α ∨ β)
   (α ⊃ β)
   (α ≡ β)
4. These are the only ways to make wffs.

Formation Rules for Wffs of **M**

1. A predicate (capital letter) followed by a constant or variable (lower-case letter) is a wff.
2. If α is a wff, so are
   (∃x)α, (∃y)α, (∃z)α, (∃w)α, (∃v)α
   (x)α, (y)α, (z)α, (w)α, (v)α
3. If α is a wff, so is ~α.
4. If α and β are wffs, then so are:
   (α · β)
   (α ∨ β)
   (α ⊃ β)
   (α ≡ β)
5. These are the only ways to make wffs.

Constructing Formal Theories

1. Specify a language.
2. Add formation rules for wffs.

To construct a formal theory, we select some of the wffs as our theorems.

3a. Specify theorems

- We can list all of our theorems: finite theories
- We can specify axioms and rules of inference.
- Proof Theory
  - Euclidean Geometry
  - Newtonian Mechanics
- Different theories can be written in the same language.

3b. Provide a semantics for the theory.

- Specify truth conditions and truth values for wffs
- Model Theory (Truth)

Goodness for Theories

- In some theories, the provable theorems are exactly the same as the true wffs.
  - *Soundness*: all the provable theorems are true
  - *Completeness*: all the truths are provable
- In more sophisticated theories, proof separates from truth
  - Gödel's first incompleteness theorem
  - There are true sentences that are not provable within any given system.
  - Model theory and proof theory provide different results.

Semantics and Proof Theory for **PL** and for **M**

- In **PL**, our semantics used truth tables.
  - Interpretations of **PL**
    - **Step 1. Assign ⊤ or ⊥ to each atomic sentence.**
      - Only finitely many (2^{26} = ~6.7 million) possible interpretations.
      - We could use infinitely many simple terms: P, P’, P’’, P’’’, P’’’’…
    - **Step 2. Assign truth values to complex propositions by combining, according to the truth table definitions, the truth values of the atomic sentences.**
  - In **M**, and the other languages of predicate logic we will study, the semantics are more complicated.
    - interpretation, satisfaction, logical truth, validity
- In proof theory, we construct a system of inference using the formal language we have specified.
  - In **PL**, our proof system was our eighteen rules of natural deduction.
    - no axioms
    - conditional and indirect proof
  - Other proof systems use axioms.

Formal system PS

- Language and wffs: those of PL
- Axiom Schemata:
  - For any wffs α, β, and γ, statements of the following forms are axioms:

    PS_1: $\alpha \supset (\beta \supset \alpha)$

    PS_2: $(\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$

    PS_3: $(\sim\alpha \supset \sim\beta) \supset (\beta \supset \alpha)$

    – Note the infinite number of axioms
- Rule of inference:
  - Modus ponens

PS and Our System of Natural Deduction

- PS and our system of natural deduction for PL are provably equivalent.
- They use equivalent languages.
- They have the same logical truths
- They are both complete.
  - All the logically true wffs are provable.
- Both systems are also sound
  - Every provable formula is logically true.

Semantics for M

- Separating the syntax of our language from its semantics allows us to treat our formal languages as completely uninterpreted.
  - Intuitively, we know what the logical operators mean.
  - But until we specify a formal interpretation, we are free to interpret them as we wish.
- Our constants and predicates and quantifiers are, as far as the syntax of our language specifies, uninterpreted.
- To look at the logical properties of the language, we construct formal semantics.
- The first step in formal semantics is to show how to provide an interpretation of the language.
- Then, we can determine the logical truths.
  - The wffs that come out as true under every interpretation.

Interpretations of M

- To define an interpretation in M, or in any of its extensions, we have to specify how to handle constants, predicates, and quantifiers.
  - We use some set theory.
  - Not in our object language, but in our meta-language.
- **Step 1**. Specify a set to serve as a domain of interpretation, or domain of quantification.
  - We can consider small finite domains

    Domain_1 = {1, 2, 3}

    Domain_2 = {Barack Obama, Hillary Clinton, and Rahm Emanuel}.
  - We can consider larger domains, like a universe of everything.
- **Step 2**. Assign a member of the domain to each constant.
- **Step 3**. Assign some set of objects in the domain to each predicate.
  - 'Ex' may stand for 'x has been elected president'
  - In Domain_1, the interpretation of 'Ex' will be empty.
  - In Domain_2, it will be {Barack Obama}.
- **Step 4**. Use the customary truth tables for the interpretation of the connectives.

Satisfaction and Truth-for-an-Interpretation

- Objects in the domain may satisfy predicates.
  - Ordered n-tuples may satisfy relations.
- A wff will be satisfiable if there are objects in the domain of quantification which satisfy the predicates indicated in the wff.
  - A universally quantified sentence is satisfied if it is satisfied by all objects in the domain.
  - An existentially quantified sentence is satisfied if it is satisfied by some object in the domain.
- A wff will be true-for-an-interpretation if all objects in the domain of quantification satisfy the predicates indicated in the wff.
- We call an interpretation on which all of a set of statements come out true a *model*.

An Interpretation of a Theory

- Theory
  1. Pa • Pb
  2. Wa • ~Wb
  3. (∃x)Px
  4. (x)Px
  5. (x)(Wx ⊃ Px)
  6. (x)(Px ⊃ Wx)
- Step 1: Specify a set to serve as a domain of interpretation, or domain of quantification.
  - Domain: {Bob Simon, Rick Werner, Katheryn Doran, Todd Franklin, Marianne Janack, Russell Marcus, Martin Shuster}
- Step 2: Assign a member of the domain to each constant.
  - a: Katheryn Doran
  - b: Bob Simon
- Step 3: Assign some set of objects in the domain to each predicate.
  - Px: {Bob Simon, Rick Werner, Katheryn Doran, Todd Franklin, Marianne Janack, Russell Marcus, Martin Shuster}
  - Wx: {Katheryn Doran, Marianne Janack}

Models in M

1. $Pa \bullet Pb$
2. $Wa \bullet {\sim}Wb$
3. $(\exists x)Px$
4. $(x)Px$
5. $(x)(Wx \supset Px)$
6. $(x)(Px \supset Wx)$

Domain: {Bob Simon, Rick Werner, Katheryn Doran, Todd Franklin, Marianne Janack, Russell Marcus, Martin Shuster}
a: Katheryn Doran
b: Bob Simon
Px: {Bob Simon, Rick Werner, Katheryn Doran, Todd Franklin, Marianne Janack, Russell Marcus, Martin Shuster}
Wx: {Katheryn Doran, Marianne Janack}

- We call an interpretation on which all of a set of given statements come out true a *model*.
- Given our interpretations of the predicates, not every sentence in our set is satisfied.
  - 1-5 are satisfied.
  - 6 is not.
- If we were to delete sentence 6 from our list, our interpretation would be a model.

Constructing a Model

- Theory
  1. (x)(Px ⊃ Qx)
  2. (∃x)(Px • Rx)
  3. (∃x)(Px • ~Rx)
  4. (∃x)(Qx • ~Rx)
  5. Pa • Pb • Qc
- **Step 1**. Specify a set to serve as a domain of interpretation, or domain of quantification.

  Domain = {Persons}
- **Step 2**. Assign a member of the domain to each constant.

  a = Barack Obama

  b = Condoleezza Rice

  c = Neytiri (from *Avatar*)
- **Step 3**. Assign some set of objects in the domain to each predicate.

  Px = {Human Beings}

  Qx = {Persons}

  Rx = {Males}
- **Step 4**. Use the customary truth tables for the interpretation of the connectives.

Logical Truth in M

- A wff of M will be logically true if it is true for every interpretation.
- For **PL**, the notion of logical truth was much simpler.
- All we had to do was look at the truth tables.
- For **M**, and even more so for **F** (full first-order logic), the notion of logical truth is just naturally complicated by the fact that we are analyzing parts of propositions.
- Here are a couple of logical truths of **M**:
  - $(x)(Px \lor {\sim}Px)$
  - $Pa \lor [(x)Px \supset Qa]$
    - Model-Theoretic Argument:
      - Consider an interpretation on which '$Pa \lor [(x)Px \supset Qa]$' is false.
      - The object assigned to 'a' will not be in the set assigned to 'Px', and there is some counterexample to $[(x)Px \supset Qa]$
      - But, any counter-example to a conditional statement has to have a true antecedent.
      - So, every object in the domain will have to be in the set assigned to 'Px'.
        - Tilt
      - So, no interpretation will make that sentence false.
      - So, '$Pa \lor [(x)Px \supset Qa]$' is logically true.
    - We could also show it proof-theoretically (using conditional or indirect proof).

Validity

- A valid argument will have to be valid under any interpretation, using any domain.
- Our proof system has given us ways to show that an argument is valid.
- But when we introduced our system of inference for **PL**, we already had a way of distinguishing the valid from the invalid arguments, using truth tables.
- In **M**, we need a corresponding method for showing that an argument is invalid.
- An invalid argument will have counter-examples, interpretations on which the premises come out true and the conclusion comes out false.