Philosophy 240
Symbolic Logic

Russell Marcus
Hamilton College
Fall 2010

Philosophy Friday #4: Three-Valued Logics
October 8, 2010

Beyond Bivalence

- A bivalent semantics is one with just two truth values: $\top$ and $\bot$.
- One non-bivalent semantics uses three truth values.
  - That name is infelicitous.
  - The difference between bivalent and three-valued interpretations comes not in the object-language logic, but in the meta-language semantics.
- Another uses more than three truth values.
  - Semantics with continuum-many truth values are called fuzzy logics.
- We will first examine eight motivations, M1-M8, for three-valued logics.
- Then, we will look at the truth tables for three three-valued logics.
- Lastly, we will briefly consider some of the problems which arise from considering a third truth-value.

M1. Mathematical sentences with unknown truth values

Goldbach's Conjecture: Every even number greater than four can be written as the sum of two odd primes.

- Goldbach's conjecture has neither been proved true nor proved false.
  - It has been verified up to very large values.
  - There are inductive arguments which make mathematicians confident that Goldbach's conjecture is true.
- Until we have a proof, we could take Goldbach's conjecture, and other unproven mathematical claims, to lack a truth value.

M2. Statements about the future

There will be a party tomorrow night on the Dunham Quad.

- Maybe there will be; maybe there will not be.
- Right now, we do not know.
- The problem of future contingents in Aristotle's *De Interpretatione*
- "In things that are not always actual there is the possibility of being and of not being; here both possibilities are open, both being and not being, and consequently, both coming to be and not coming to be" (*De Interpretatione* §9.19a9-13).
- "It is necessary for there to be or not to be a sea-battle tomorrow; but it is not necessary for a sea-battle to take place tomorrow, nor for one not to take place - though it is necessary for one to take place or not to take place" (*De Interpretatione* §9.19a30-33).

Aristotle's View

1. Either there will be a sea-battle tomorrow or there will not be a sea-battle tomorrow.
2. There will be a sea-battle tomorrow.
3. There will not be a sea-battle tomorrow.

- Call 1 true, indeed necessarily true.
- Withhold truth values from 2 and 3.
- If 2 and 3 are not true, and we only have two truth values, then they must be false.
- If 2 and 3 are false, we should be willing to assert their negations.
  - 2'. It is not the case that there will be a sea-battle tomorrow.
  - 3'. It is not the case that there will not be a sea-battle tomorrow.
- 2' and 3' represent our acknowledgment of the contingency of the event.
- But, taken together, 2', and 3' form a contradiction.
  - $\sim P \bullet \sim\sim P$
- If we have a third truth-value, we can assert both 2 and 3 without contradiction.

M3. Failure of presupposition

- Consider the following two claims:
  1. The king of America is bald.
  2. The king of America is not bald.
- Neither 1 nor 2 are true propositions.
- But, 2 looks like the negation of 1.
- That is, if we regiment 1 as ‘P’, we should regiment 2 as ‘~P’.
- In a bivalent logic, since ‘P’ is not true, we must call it false, since we have only two truth values.
- Assigning the value ‘false’ to ‘P’ means that ‘~P’ should be assigned ‘true’.
- Uh-oh.

Other Failures of Presupposition

- The woman on the moon is six feet tall.
- The rational square root of three is less than two.
- When did you stop beating your wife?
- One response to the problem of presupposition failure in propositions is to call such propositions neither true nor false.

M4. Nonsense

Quadruplicity drinks procrastination.
Colorless green ideas sleep furiously.

- The syntax of a formal language tells us whether a string of symbols of the language is a wff.
- The correlate of syntax, in natural language, is grammaticality.
- But, not all grammatical sentences are sensible.
- We might consider some grammatical but nonsensical sentences to be truth-valueless.

M5. Programming needs

- Computer circuits are just series of switches which can either be open or closed.
- To interpret the state of the circuit, we take a closed switch to be true and an open switch to be false.
- We might want to leave the values of some variables unassigned, during a process.
- For example, we might want to know how a system works without knowing whether a given switch is open or closed.
- Consider an on-line form which must be completed by a user.
  - We might want to allow a user to leave certain entries in the form blank.
  - Or, we might want to make sure that the program does not crash when those entries are left blank.

M6. Semantic paradoxes

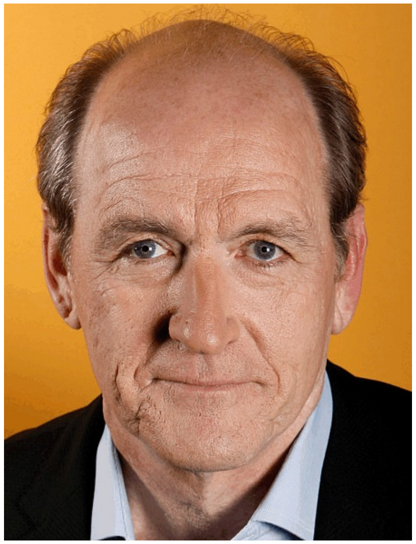
E. This sentence is false.

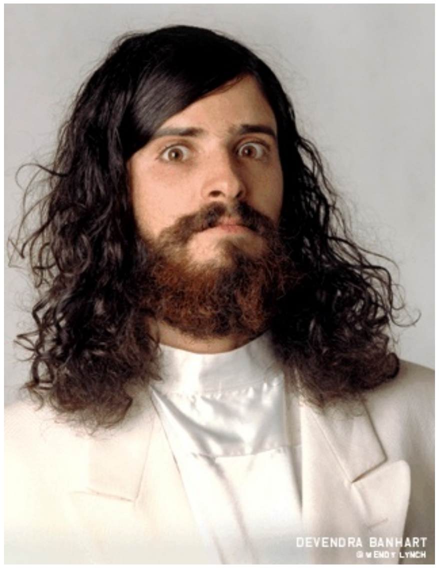
- If E is true, then it is false, which makes it true, which makes it false...
- E seems to lack a definite truth value, even though it is a perfectly well-formed sentence.
- Since assigning ⊤ or ⊥ to E leads to a contradiction, we might assign it a third truth value.

M7. The Paradoxes of the Material Conditional

- $\alpha \supset (\beta \supset \alpha)$
- $\sim\alpha \supset (\alpha \supset \beta)$
- $(\alpha \supset \beta) \vee (\beta \supset \alpha)$

M8. Vagueness





Three-Valued logics

- Bivalent semantics is also called classical semantics
- For a third truth value, we could introduce an unknown or indeterminate value.
- Remember, the idea is that we can ascribe ı to sentences which lack a clear truth value.
- There are two options for how to deal with unknown or indeterminate truth values in the new semantics.
  - Any indeterminacy among component propositions creates indeterminacy in the whole.
  - Ascribe truth values to as many formulas as possible, despite the indeterminate truth values.
- We proceed to look at three different three-valued semantics.
  - 1. The rules for each;
  - 2. How the new rules affect the logical truths (tautologies); and
  - 3. How the new rules affect the allowable inferences (valid arguments).

Bochvar semantics (B)

Any indeterminacy infects the whole.

α	~α
⊤	⊥
ι	ι
⊥	⊤

α	•	β
⊤	⊤	⊤
⊤	ι	ι
⊤	⊥	⊥
ι	ι	⊤
ι	ι	ι
ι	ι	⊥
⊥	⊥	⊤
⊥	ι	ι
⊥	⊥	⊥

α	∨	β
⊤	⊤	⊤
⊤	ι	ι
⊤	⊤	⊥
ι	ι	⊤
ι	ι	ι
ι	ι	⊥
⊥	⊤	⊤
⊥	ι	ι
⊥	⊥	⊥

α	⊃	β
⊤	⊤	⊤
⊤	ι	ι
⊤	⊥	⊥
ι	ι	⊤
ι	ι	ι
ι	ι	⊥
⊥	⊤	⊤
⊥	ι	ι
⊥	⊤	⊥

Two Classical Tautologies

P	⊃	P
⊤	⊤	⊤
ι	ι	ι
⊥	⊤	⊥

P	⊃	(Q	⊃	P)
⊤	⊤	⊤	⊤	⊤
⊤	ι	ι	ι	⊤
⊤	⊤	⊥	⊤	⊤
ι	ι	⊤	ι	ι
ι	ι	ι	ι	ι
ι	ι	⊥	ι	ι
⊥	⊤	⊤	⊥	⊥
⊥	ι	ι	ι	⊥
⊥	⊤	⊥	⊤	⊥

- These classical tautologies, and all others, do not come out false on any line on Bochvar semantics.
- But, they do not come out as true on every line.
- This result is generally undesirable, since most classical tautologies seem pretty solid.
- The paradoxes of material implication are eliminated.

‘P / Q ∨ P’

P	//	Q	∨	P
⊤		⊤	⊤	⊤
⊤		⊥	⊤	⊤
⊥		⊤	⊤	⊥
⊥		⊥	⊥	⊥

- Under classical semantics, this argument is valid.
- Under Bochvar semantics, the second row is a counterexample!

P	//	Q	∨	P
⊤		⊤	⊤	⊤
⊤		ι	ι	⊤
⊤		⊥	⊤	⊤
ι		⊤	ι	ι
ι		ι	ι	ι
ι		⊥	ι	ι
⊥		⊤	⊤	⊥
⊥		ι	ι	⊥
⊥		⊥	⊥	⊥

Options

- Bochvar semantics seems to throw the baby out with the bath water.
- We could re-define some terms.
- We could take ‘tautology’ as a statement which never comes out as false.
- Validity
  - A valid argument is one for which there is no row in which the premises are true and the conclusion is false.
  - An alternative: a valid argument is one for which there is no row in which the premises are true and the conclusion is not true.
- Redefining ‘tautology’ and ‘validity’ in this way weakens the concepts, making them less useful.
- Quine: three-valued logics are “deviant.”
- They change the subject.
- Is a row in which the premises are true and the conclusion is indeterminate a counterexample?

Alternatives to Bochvar

- Why should we consider the disjunction of a true statement with one of indeterminate truth value to be undetermined?
- Why should we consider the conditional with an antecedent of indeterminate truth value to itself be of indeterminate truth value, if the consequent is true?
- Whatever other value we can assign the variables with unknown truth value, both sentences will turn out to be true.
- Kleene’s semantics leaves fewer rows unknown.

Kleene semantics (K3)

P	~P
⊤	⊥
ι	ι
⊥	⊤

P	•	Q
⊤	⊤	⊤
⊤	ι	ι
⊤	⊥	⊥
ι	ι	⊤
ι	ι	ι
ι	⊥	⊥
⊥	⊥	⊤
⊥	⊥	ι
⊥	⊥	⊥

P	∨	Q
⊤	⊤	⊤
⊤	⊤	ι
⊤	⊤	⊥
ι	⊤	⊤
ι	ι	ι
ι	ι	⊥
⊥	⊤	⊤
⊥	ι	ι
⊥	⊥	⊥

P	⊃	Q
⊤	⊤	⊤
⊤	ι	ι
⊤	⊥	⊥
ι	⊤	⊤
ι	ι	ι
ι	ι	⊥
⊥	⊤	⊤
⊥	⊤	ι
⊥	⊤	⊥

$P \supset P$

P	⊃	P
⊤	⊤	⊤
ι	**ι**	ι
⊥	⊤	⊥

Bochvar and Kleene
yield the same results

$P \supset (Q \supset P)$

P	$\supset$	(Q	$\supset$	P)
$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	ι	ι	ι	$\top$
$\top$	$\top$	$\bot$	$\top$	$\top$
ι	ι	$\top$	ι	ι
ι	ι	ι	ι	ι
ι	ι	$\bot$	ι	ι
$\bot$	$\top$	$\top$	$\bot$	$\bot$
$\bot$	ι	ι	ι	$\bot$
$\bot$	$\top$	$\bot$	$\top$	$\bot$

Bochvar

P	$\supset$	(Q	$\supset$	P)
$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\top$	ι	$\top$	$\top$
$\top$	$\top$	$\bot$	$\top$	$\top$
ι	ι	$\top$	ι	ι
ι	ι	ι	ι	ι
ι	$\top$	$\bot$	$\top$	ι
$\bot$	$\top$	$\top$	$\bot$	$\bot$
$\bot$	$\top$	ι	ι	$\bot$
$\bot$	$\top$	$\bot$	$\top$	$\bot$

Kleene

K3 versus Bochvar

- Many classically logical truths still do not come out as tautologous.
- But more rows are completed in K3.
- Lukasiewicz tried to preserve the tautologies.

Lukasiewicz semantics (L3)

P	~P
$\top$	$\bot$
ι	ι
$\bot$	$\top$

P	•	Q
$\top$	$\top$	$\top$
$\top$	ι	ι
$\top$	$\bot$	$\bot$
ι	ι	$\top$
ι	ι	ι
ι	$\bot$	$\bot$
$\bot$	$\bot$	$\top$
$\bot$	$\bot$	ι
$\bot$	$\bot$	$\bot$

P	$\vee$	Q
$\top$	$\top$	$\top$
$\top$	$\top$	ι
$\top$	$\top$	$\bot$
ι	$\top$	$\top$
ι	ι	ι
ι	ι	$\bot$
$\bot$	$\top$	$\top$
$\bot$	ι	ι
$\bot$	$\bot$	$\bot$

P	$\supset$	Q
$\top$	$\top$	$\top$
$\top$	ι	ι
$\top$	$\bot$	$\bot$
ι	$\top$	$\top$
ι	$\top$	ι
ι	ι	$\bot$
$\bot$	$\top$	$\top$
$\bot$	$\top$	ι
$\bot$	$\top$	$\bot$

The only difference between K3 and L3 is in the fifth row of the truth table for the conditional.

Top-Down Arguments

- One might wonder how we could justify calling a conditional with indeterminate truth values in both the antecedent and consequent true.
- What if the antecedent turns out true and the consequent turns out false?
- We motivated Bochvar and Kleene semantics by looking at each cell in the truth table.
- We can motivate L3 by looking at the results.

P ⊃ P

P	⊃	P
⊤	⊤	⊤
ι	ι	ι
⊥	⊤	⊥

Bochvar/
Kleene

P	⊃	P
⊤	⊤	⊤
ι	⊤	ι
⊥	⊤	⊥

L3

P ⊃ (Q ⊃ P)

P	⊃	(Q	⊃	P)
⊤	⊤	⊤	⊤	⊤
⊤	ι	ι	ι	⊤
⊤	⊤	⊥	⊤	⊤
ι	ι	⊤	ι	ι
ι	ι	ι	ι	ι
ι	ι	⊥	ι	ι
⊥	⊤	⊤	⊥	⊥
⊥	ι	ι	ι	⊥
⊥	⊤	⊥	⊤	⊥

Bochvar

P	⊃	(Q	⊃	P)
⊤	⊤	⊤	⊤	⊤
⊤	⊤	ι	⊤	⊤
⊤	⊤	⊥	⊤	⊤
ι	ι	⊤	ι	ι
ι	ι	ι	ι	ι
ι	⊤	⊥	⊤	ι
⊥	⊤	⊤	⊥	⊥
⊥	⊤	ι	ι	⊥
⊥	⊤	⊥	⊤	⊥

K3

P	⊃	(Q	⊃	P)
⊤	⊤	⊤	⊤	⊤
⊤	⊤	ι	⊤	⊤
⊤	⊤	⊥	⊤	⊤
ι	⊤	⊤	ι	ι
ι	⊤	ι	⊤	ι
ι	⊤	⊥	⊤	ι
⊥	⊤	⊤	⊥	⊥
⊥	⊤	ι	ι	⊥
⊥	⊤	⊥	⊤	⊥

L3

L3 and Tautologies

- In L3, we retain many, though not all, of the classical tautologies!
- ‘P ∨ ~P’ is still not a tautology.

L3 and Tautologies

- In L3, we retain many, though not all, of the classical tautologies!
- ‘P ∨ ~P’ is still not a tautology.
  - Some folks would like to abandon that law, anyway.
  - Indeed, three-valued logics were motivated largely by rejection of that law.
- We need not give up classical logical truths to have a three-valued logic.
- The fewer changes we make to the set of logical truths, the less “deviant” the logic is.
- But, the semantics which allows us to retain these logical truths may not be as pretty as we would like.

P	∨	~	P
⊤	⊤	⊥	⊤
ι	**ι**	ι	ι
⊥	⊤	⊤	⊥

P / Q ∨ P

P	//	Q	∨	P
⊤		⊤	⊤	⊤
⊤		ι	ι	⊤
⊤		⊥	⊤	⊤
ι		⊤	ι	ι
ι		ι	ι	ι
ι		⊥	ι	ι
⊥		⊤	⊤	⊥
⊥		ι	ι	⊥
⊥		⊥	⊥	⊥

Bochvar: counter-example in Row 2

P	//	Q	∨	P
⊤		⊤	⊤	⊤
⊤		ι	ι	⊤
⊤		⊥	⊤	⊤
ι		⊤	ι	ι
ι		ι	ι	ι
ι		⊥	ι	ι
⊥		⊤	⊤	⊥
⊥		ι	ι	⊥
⊥		⊥	⊥	⊥

Kleene: valid - no counter-example

P	//	Q	∨	P
⊤		⊤	⊤	⊤
⊤		ι	⊤	⊤
⊤		⊥	⊤	⊤
ι		⊤	⊤	ι
ι		ι	ι	ι
ι		⊥	ι	ι
⊥		⊤	⊤	⊥
⊥		ι	ι	⊥
⊥		⊥	⊥	⊥

Lukasiewicz: valid - no counter-example

A Third Value, or a Missing Value?

- Assigning a truth-value of ‘unknown’ involves a conceptual confusion.
- ‘Unknown’ may not be a third truth value, but merely the lack of a truth value.
- Instead of filling in such cells in the truth table, we should just leave them blank.
- Leaving certain cells of the truth table blank is part of what is called the truth-value gap approach.
- Faced with truth-value gaps, or partial valuations, the logician may consider something called a supervaluation.
- A supervaluation considers the different ways to complete partial valuations, and classifies formulas and arguments according to the possibilities for completion.

Does Three-Valued Logic Solve The Motivating Problems?

M1. Mathematical sentences with unknown truth values

M2. Statements about the future

M3. Failure of presupposition

M4. Nonsense

M5. Programming needs

M6. Semantic paradoxes

M7. The paradoxes of the material conditional

M8. Vagueness

Maxim of Minimum Mutilation: Make the least disruptive change in your theory.

Epistemic Confusion

M1 and M2

- We can say that Goldbach's conjecture is either true or false, but we just do not know which.
- We can say that either there will be a party at Dunham tomorrow, or there will not.
- In both cases, we can blame ourselves, rather than the world, for our not knowing the truth value.
- We need not ascribe a deep problem to truth values.
  - Such sentences have truth values.
  - We just do not know them.
- Four-dimensionalism
  - We add a time-stamp to all our claims.
  - We can take a God's-eye point of view for time problems.
  - A statement about the future is true if it ends up true at the time.

Definite Descriptions and the Failures of Presupposition

M3

- 1: The woman on the moon is six feet tall.
- 2: There is a woman on the moon and she is six feet tall.
- Sentence 2 has the form 'P • Q'.
- 'P' is false, so 'P • Q' is false.
  - 3. The woman on the moon is not six feet tall.
  - 4: There is a woman on the moon and she is not six feet tall.
- In both cases, P is false, so the account of the falsity of both sentences D and E can be the same.
- We lose the motivation for introducing a third truth value.

Denying that a Sentence Expresses a Proposition

M4, nonsense, and M8, vagueness

- We can claim that just as some strings of letters do not form words, and some strings of words do not form sentences, some grammatical sentences do not express propositions.
- This would be the same as to call them meaningless.

The Strengthened Liar

M6

- Bochvar hoped that his semantics would solve the problems of the semantic paradoxes.
- The liar sentence can be given a truth value in Bochvar semantics without paradox.
- A: A is untrue
  - Suppose A is true.
  - Thus, A is untrue.
  - But then A turns out to be true (because it says that A is untrue).

Others

- The needs of computer programmers (M5) shouldn't motivate our choice of logic, though there's no reason not to develop logics for those purposes.
- We can live the the paradoxes of the material conditional (M6).
- The problems of the conditional is, as we've seen, not a logical problem.
- Like the problem of vagueness, it is too broad to be solved by adopting a third truth value.
- Another motivation: quantum physics
- That's a good paper topic!

Philosophy 240
Symbolic Logic

Russell Marcus
Hamilton College
Fall 2010

Philosophy Friday #4: Three-Valued Logics
October 8, 2010