Hamilton College
Russell Marcus
rmarcus1@hamilton.edu

Class 19 - October 8
Philosophy Friday #3: Three-Valued Logics

**O. Introduction**

We have been both working with our system of propositional logic, in the object language, and interpreting our system, doing semantics in the meta-language.
The semantics of propositional logic mainly consist in the assignments of truth values to propositional variables.
We represent these assignments using the truth tables.
For the most part, in this course, we use a bivalent interpretation of our logic.
A bivalent semantics is one with just two truth values: ⊤ and ⊥.
For various reasons, some logicians have constructed semantics for propositional logic with more than two truth values.
These semantics divide into two classes.
The first type of non-bivalent semantics uses three truth values.
The second type uses more than three truth values.
Most of the interpretations which use more than three truth values use infinitely many, indeed continuum-many, truth values.
Semantics with continuum-many truth values are called fuzzy logics.

Three-valued interpretations are generally called three-valued logics.
That name is infelicitous, since the difference between bivalent and three-valued interpretations comes not in the object-language logic but in the meta-language semantics.
We will first examine eight motivations, M1-M8, for three-valued logics.
We will go quickly through some of these motivations, especially the ones which we will discuss, or have discussed, at other times in this course.
Then, we will look at the truth tables for three three-valued logics.
Lastly, we will briefly consider some of the problems which arise from considering a third truth-value.

**I. Eight Motivations for Three-Valued Logics**

**M1. Mathematical sentences with unknown truth values**

Consider:

A. Every even number greater than four can be written as the sum of two odd primes.

A is called Goldbach's conjecture, though it seems that Euler actually formulated it in response to a weaker hypothesis raised by Goldbach in 1742.
Goldbach's conjecture has neither been proved true nor proved false.
It has been verified up to very large values.
There are, also, inductive arguments which make mathematicians pretty confident that Goldbach's conjecture is true.

But, many smart mathematicians have tried and failed to devise a proof.

We might take Goldbach's conjecture to be neither true nor false.
We might do so, especially, if we think that mathematics is constructed, rather than discovered.
If and when some one proves it, or its negation, then we could apply a truth value to the proposition.
Until we have a proof, we could take Goldbach's conjecture, and other unproven mathematical claims, to lack a truth value.

**M2. Statements about the future**

Consider:

B. There will be a party tomorrow night on the Dunham Quad.

Maybe there will be; maybe there will not be.
Right now, we can not assign a truth value to B.
The problem in B, generally labeled the problem of future contingents, may be found in Aristotle's *De Interpretatione* regarding a sea battle.
Since B is contingent, we can, at this moment, neither assert its truth nor its falsity.

> In things that are not always actual there is the possibility of being and of not being; here both possibilities are open, both being and not being, and consequently, both coming to be and not coming to be (*De Interpretatione* §9.19a9-13).

We know that one of the two truth values will apply, eventually.
But, right now, it seems to lack a truth value.

> It is necessary for there to be or not to be a sea-battle tomorrow; but it is not necessary for a sea-battle to take place tomorrow, nor for one not to take place - though it is necessary for one to take place or not to take place (*De Interpretatione* §9.19a30-33).

If the claim that there will be a sea-battle tomorrow had a truth value now, then the event would not be contingent; it would already be determined.
Since the future is not determined, the truth values of statements about the future should also be undetermined.

We can understand Aristotle's claim better by considering the following three claims.

C. Either there will be a sea-battle tomorrow or there will not be a sea-battle tomorrow.
D. There will be a sea-battle tomorrow.
E. There will not be a sea-battle tomorrow.

Aristotle wants to call C true, indeed necessarily true, while withholding truth values from D and E.
If D and E are not true, and we only have two truth values, then they must be false.
If D and E are false, we should be willing to assert their negations.

D'. It is not the case that there will be a sea-battle tomorrow.
E'. It is not the case that there will not be a sea-battle tomorrow.

D' and E' represent our acknowledgment of the contingency of the event.
But, taken together, D', and E' form a contradiction.

F. ~P • ~~P

| ~ | P | • | ~ | ~ | P |
|---|---|---|---|---|---|
| ⊥ | ⊤ | ⊥ | ⊤ | ⊥ | ⊤ |
| ⊤ | ⊥ | ⊥ | ⊥ | ⊤ | ⊥ |

If we have a third truth-value, we can assert both D' and E' without contradiction.
In a three-valued logic, denying that a statement is true does not entail that it is false.
It could be neither true nor false.

**M3. Failure of presupposition**

Consider the following two claims:

G. The king of America is bald.
H: The king of America is not bald.

Neither G nor H are true propositions.
But, H looks like the negation of G.
That is, if we regiment G as 'P', we should regiment H as '~P'.
In a bivalent logic, since 'P' is not true, we must call it false, since we have only two truth values.
Assigning the value 'false' to 'P' means that '~P' should be assigned 'true'.
Uh-oh.

The problem is that in this case we want both a proposition and its negation to be false.
But in a bivalent logic, the negation of a false proposition is a true proposition.
Thus, we can never, in a bivalent logic, deny both a statement and its negation, as we wish to do with G and H.

We think that G and H are both false because they both contain a false presupposition.
The following sentences all contain a failure of presupposition.

I: The woman on the moon is six feet tall.
J: The rational square root of three is less than two.
K: When did you stop beating your wife?

Sentence K is not declarative, but also contains a failure of presupposition.
One response to the problem of presupposition failure in propositions is to call such propositions neither true nor false.

**M4. Nonsense**

The distinction between syntax and semantics can be a motivation to adopt a third truth-value.
The syntax of a formal language tells us whether a string of symbols of the language is a wff.
The correlate of syntax, in natural language, is grammaticality.
But, not all grammatical sentences are sensible.
We might consider some grammatical but nonsensical sentences to be truth-valueless.

L. Quadruplicity drinks procrastination. (From Bertrand Russell)
M. Colorless green ideas sleep furiously. (From Noam Chomsky)

As far as the syntax of English is concerned, L and M are well-formed.
But, their well-formedness does not entail that we can assign truth values to them.

**M5. Programming needs**

Logic is essential to the program of a computer.
At the most basic level, computer circuits are just series of switches which can either be open or closed.
To represent the circuit, we use logic.
Roughly, electrons can pass through a switch if it is closed and can not pass through if it is open.
To interpret the state of the circuit, we take a closed switch to be true and an open switch to be false.
We might want to leave the values of some variables unassigned, during a process.
For example, we might want to know how a system works without knowing whether a given switch is open or closed.

On a higher level, consider an on-line form which must be completed by a user.
We might want to allow a user to leave certain entries in the form blank.
Or, we might want to make sure that the program does not crash when those entries are left blank.
So, we might want a logic which uses a third value.
M5 is merely a pragmatic motivation, rather than a philosophical one.
It is, though, a reason to explore the technical merits of three-valued logics.
Then, we can wonder about the philosophical motivations and applications.

**M6. Semantic paradoxes**

There are a variety of semantic paradoxes.
The most famous is called the liar.

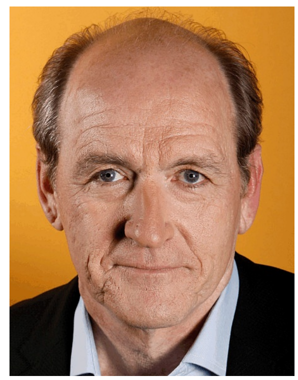
N. This sentence is false.

N is an example of a paradoxical sentence.
If N is true, then it is false, which makes it true, which makes it false...
N seems to lack a definite truth value, even though it is a perfectly well-formed sentence.
It is often called Epimenides' paradox.
Epimenides was a Cretan to whom the statement that all Cretans are liars is attributed.
Since assigning $\top$ or $\bot$ to N leads to a contradiction, we might assign it a third truth value.

**M7. The paradoxes of the material conditional**
See elsewhere for an extended discussion.

**M8. Vagueness**

Many predicates admit of borderline, or vague, cases.
Consider baldness.
There are paradigm instances of baldness which are incontrovertible.
There are also paradigm instances of non-baldness.
But, there are also cases in which we don’t know what to say.

O. Tyra Banks is bald.
P. Devendra Banhart is bald.
Q. Richard Jenkins is bald.

O is true; P is false.
But between Tyra Banks and Devendra Banhart, there is a penumbra.
We could, if we wish, give Q a third truth value.
More compellingly, we could apply a logic in which there are infinitely many truth values to this case.
To do so, we could assign a value of 0 to O, and a value of 1 to P.
Then, we can assign any real-number between 0 and 1, say .3, to Q.
I will not pursue this proposal, called fuzzy logic, here.

**II. Three-valued logics**

The rules for determining truth values of formulas in a logic are called the semantics.
We provided semantics for propositional logic by constructing truth tables.
Since we used only two values, true and false, our semantics is called two-valued.
Our two-valued semantics is also called classical semantics
If we want to adopt a third truth value, which we might call unknown, or indeterminate, we must revise all the truth tables.
I will call the third truth value indeterminate, and use the Greek letter ι to indicate it.

Remember, the idea is that we can ascribe ι to sentences which lack a clear truth value.
There are two options for how to deal with unknown or indeterminate truth values in the new semantics.
First, one could claim that any indeterminacy among component propositions creates indeterminacy in the whole.
This the principle underlying Bochvar's semantics.
Second, one could try to ascribe truth values to as many formulas as possible, despite the indeterminate truth values.

We proceed to look at three different three-valued semantics.
We will look at:

1. The rules for each;
2. How the new rules affect the logical truths (tautologies); and
3. How the new rules affect the allowable inferences (valid arguments).

**III. Bochvar semantics (B)**

| $\alpha$ | $\sim\alpha$ |
|---|---|
| $\top$ | $\bot$ |
| ι | ι |
| $\bot$ | $\top$ |

| $\alpha$ | • | $\beta$ |
|---|---|---|
| $\top$ | $\top$ | $\top$ |
| $\top$ | ι | ι |
| $\top$ | $\bot$ | $\bot$ |
| ι | ι | $\top$ |
| ι | ι | ι |
| ι | ι | $\bot$ |
| $\bot$ | $\bot$ | $\top$ |
| $\bot$ | ι | ι |
| $\bot$ | $\bot$ | $\bot$ |

| $\alpha$ | $\vee$ | $\beta$ |
|---|---|---|
| $\top$ | $\top$ | $\top$ |
| $\top$ | ι | ι |
| $\top$ | $\top$ | $\bot$ |
| ι | ι | $\top$ |
| ι | ι | ι |
| ι | ι | $\bot$ |
| $\bot$ | $\top$ | $\top$ |
| $\bot$ | ι | ι |
| $\bot$ | $\bot$ | $\bot$ |

| $\alpha$ | $\supset$ | $\beta$ |
|---|---|---|
| $\top$ | $\top$ | $\top$ |
| $\top$ | ι | ι |
| $\top$ | $\bot$ | $\bot$ |
| ι | ι | $\top$ |
| ι | ι | ι |
| ι | ι | $\bot$ |
| $\bot$ | $\top$ | $\top$ |
| $\bot$ | ι | ι |
| $\bot$ | $\top$ | $\bot$ |

For simplicity, we ignore the biconditional, which is definable in terms of the other connectives anyway.
In Bochvar semantics, no classical tautologies come out as tautologies.

Consider 'P ⊃ P', under Bochvar semantics:

| P | ⊃ | P |
|---|---|---|
| ⊤ | ⊤ | ⊤ |
| ι | ι | ι |
| ⊥ | ⊤ | ⊥ |

Or, 'P ⊃ (Q ⊃ P)' under Bochvar:

| P | ⊃ | (Q | ⊃ | P) |
|---|---|---|---|---|
| ⊤ | ⊤ | ⊤ | ⊤ | ⊤ |
| ⊤ | ι | ι | ι | ⊤ |
| ⊤ | ⊤ | ⊥ | ⊤ | ⊤ |
| ι | ι | ⊤ | ι | ι |
| ι | ι | ι | ι | ι |
| ι | ι | ⊥ | ι | ι |
| ⊥ | ⊤ | ⊤ | ⊥ | ⊥ |
| ⊥ | ι | ι | ι | ⊥ |
| ⊥ | ⊤ | ⊥ | ⊤ | ⊥ |

These classical tautologies, and all others, do not come out false on any line on Bochvar semantics.
But, they do not come out as true on every line.
This result is generally undesirable, since the classical tautologies seem pretty solid.
Tautologies are also known as logical truths.
They are the theorems of the logic.

Some people think that some of the classical tautologies are suspect anyway.
In particular, 'P ⊃ (Q ⊃ P)' seems counter-intuitive.
It is sometimes called a paradox of material implication.
Other systems of logic, called relevance logics, attempt to keep most classical logical truths, but eliminate the paradoxes of material implication.
Unfortunately, Bochvar semantics seems to throw the baby out with the bath water, in eliminating all classical tautologies.

One solution to the problem of losing logical truths in Bochvar semantics would be to redefine 'tautology' as a statement which never comes out as false.
Redefining 'tautology' in this way, though, weakens the concept, making it less useful.
Quine, in *Philosophy of Logic*, Chapter 6, calls three-valued logics "deviant", and urges that they

constitute a change of topic, rather than an improvement of logic.
Now, consider what Bochvar semantics does to validity.
We defined a valid argument as one for which there is no row in which the premises are true and the conclusion is false.
We could have defined a valid argument as one for which there is no row in which the premises are true and the conclusion is not true.
Classically, these two definitions are equivalent.
But, in three-valued semantics, they cleave.
If we take a row in which the premises are true and the conclusion is indeterminate as a counterexample to an argument, as Bochvar did, then some classically valid inferences come out invalid.

Consider: ‘P / Q ∨ P’
Under classical semantics, this argument is valid.

| P | // | Q | ∨ | P |
|---|---|---|---|---|
| ⊤ | | ⊤ | ⊤ | ⊤ |
| ⊤ | | ⊥ | ⊤ | ⊤ |
| ⊥ | | ⊤ | ⊤ | ⊥ |
| ⊥ | | ⊥ | ⊥ | ⊥ |

Now, look at the same argument under Bochvar semantics:

| P | // | Q | ∨ | P |
|---|---|---|---|---|
| ⊤ | | ⊤ | ⊤ | ⊤ |
| ⊤ | | ι | **ι** | ⊤ |
| ⊤ | | ⊥ | ⊤ | ⊤ |
| **ι** | | ⊤ | **ι** | ι |
| **ι** | | ι | **ι** | ι |
| **ι** | | ⊥ | **ι** | ι |
| ⊥ | | ⊤ | ⊤ | ⊥ |
| ⊥ | | ι | **ι** | ⊥ |
| ⊥ | | ⊥ | ⊥ | ⊥ |

The second row is now a counterexample!

Bochvar semantics proceeds on the presupposition that any indeterminacy infects the whole.
It thus leaves the truth values of many formulas undetermined.

But, we might be able to fill in some of the holes.
That is, why should we consider the disjunction of a true statement with one of indeterminate truth value to be undetermined?
Or, why should we consider the conditional with an antecedent of indeterminate truth value to itself be of indeterminate truth value, if the consequent is true?
Whatever other value we can assign the variables with unknown truth value, both sentences will turn out to be true.
Kleene's semantics leaves fewer rows unknown.
(The semantics that follows is sometimes called strong Kleene, or K3) to distinguish from Bochvar, which is sometimes called weak Kleene.)

**IV. Kleene semantics (K3)**

| P | ~P |
|---|---|
| ⊤ | ⊥ |
| ι | ι |
| ⊥ | ⊤ |

| P | • | Q |
|---|---|---|
| ⊤ | ⊤ | ⊤ |
| ⊤ | ι | ι |
| ⊤ | ⊥ | ⊥ |
| ι | ι | ⊤ |
| ι | ι | ι |
| ι | ⊥ | ⊥ |
| ⊥ | ⊥ | ⊤ |
| ⊥ | ⊥ | ι |
| ⊥ | ⊥ | ⊥ |

| P | ∨ | Q |
|---|---|---|
| ⊤ | ⊤ | ⊤ |
| ⊤ | ⊤ | ι |
| ⊤ | ⊤ | ⊥ |
| ι | ⊤ | ⊤ |
| ι | ι | ι |
| ι | ι | ⊥ |
| ⊥ | ⊤ | ⊤ |
| ⊥ | ι | ι |
| ⊥ | ⊥ | ⊥ |

| P | ⊃ | Q |
|---|---|---|
| ⊤ | ⊤ | ⊤ |
| ⊤ | ι | ι |
| ⊤ | ⊥ | ⊥ |
| ι | ⊤ | ⊤ |
| ι | ι | ι |
| ι | ι | ⊥ |
| ⊥ | ⊤ | ⊤ |
| ⊥ | ⊤ | ι |
| ⊥ | ⊤ | ⊥ |

Kleene semantics has a certain intuitiveness.
But, in order to compare Bochvar to Kleene properly, we should look at the differences on logical truths and inference patterns.

Consider the same two tautologies, 'P ⊃ P' and 'P ⊃ (Q ⊃ P)' under Kleene semantics:

| P | ⊃ | P |
|---|---|---|
| ⊤ | ⊤ | ⊤ |
| ι | ι | ι |
| ⊥ | ⊤ | ⊥ |

| P | ⊃ | (Q | ⊃ | P) |
|---|---|---|---|---|
| ⊤ | ⊤ | ⊤ | ⊤ | ⊤ |
| ⊤ | ⊤ | ι | ⊤ | ⊤ |
| ⊤ | ⊤ | ⊥ | ⊤ | ⊤ |
| ι | ι | ⊤ | ι | ι |
| ι | ι | ι | ι | ι |
| ι | ⊤ | ⊥ | ⊤ | ι |
| ⊥ | ⊤ | ⊤ | ⊥ | ⊥ |
| ⊥ | ⊤ | ι | ι | ⊥ |
| ⊥ | ⊤ | ⊥ | ⊤ | ⊥ |

While many more of the rows are completed, the statements still do not come out as tautologous, under the classical definition of 'tautology'.
Lukasiewicz, who first investigated three-valued logics, tried to preserve the tautologies.
There is only one difference between Kleene semantics and Lukasiewicz semantics.

**V. Lukasiewicz semantics (L3)**

| P | ~P |
|---|---|
| ⊤ | ⊥ |
| ι | ι |
| ⊥ | ⊤ |

| P | • | Q |
|---|---|---|
| ⊤ | ⊤ | ⊤ |
| ⊤ | ι | ι |
| ⊤ | ⊥ | ⊥ |
| ι | ι | ⊤ |
| ι | ι | ι |
| ι | ⊥ | ⊥ |
| ⊥ | ⊥ | ⊤ |
| ⊥ | ⊥ | ι |
| ⊥ | ⊥ | ⊥ |

| P | ∨ | Q |
|---|---|---|
| ⊤ | ⊤ | ⊤ |
| ⊤ | ⊤ | ι |
| ⊤ | ⊤ | ⊥ |
| ι | ⊤ | ⊤ |
| ι | ι | ι |
| ι | ι | ⊥ |
| ⊥ | ⊤ | ⊤ |
| ⊥ | ι | ι |
| ⊥ | ⊥ | ⊥ |

| P | ⊃ | Q |
|---|---|---|
| ⊤ | ⊤ | ⊤ |
| ⊤ | ι | ι |
| ⊤ | ⊥ | ⊥ |
| ι | ⊤ | ⊤ |
| ι | ⊤ | ι |
| ι | ι | ⊥ |
| ⊥ | ⊤ | ⊤ |
| ⊥ | ⊤ | ι |
| ⊥ | ⊤ | ⊥ |

One might wonder how we might justify calling a conditional with indeterminate truth values in both the antecedent and consequent true.
For, what if the antecedent turns out true and the consequent turns out false?

Put that worry aside, and look at what this one small change does:

| P | $\supset$ | P |
|---|---|---|
| $\top$ | $\top$ | $\top$ |
| $\iota$ | $\top$ | $\iota$ |
| $\bot$ | $\top$ | $\bot$ |

| P | $\supset$ | (Q | $\supset$ | P) |
|---|---|---|---|---|
| $\top$ | $\top$ | $\top$ | $\top$ | $\top$ |
| $\top$ | $\top$ | $\iota$ | $\top$ | $\top$ |
| $\top$ | $\top$ | $\bot$ | $\top$ | $\top$ |
| $\iota$ | $\top$ | $\top$ | $\iota$ | $\iota$ |
| $\iota$ | $\top$ | $\iota$ | $\top$ | $\iota$ |
| $\iota$ | $\top$ | $\bot$ | $\top$ | $\iota$ |
| $\bot$ | $\top$ | $\top$ | $\bot$ | $\bot$ |
| $\bot$ | $\top$ | $\iota$ | $\iota$ | $\bot$ |
| $\bot$ | $\top$ | $\bot$ | $\top$ | $\bot$ |

Voila!
We retain many of the classical tautologies!
In fact, we do not get all classical tautologies.
Consider the law of excluded middle, 'P $\vee$ ~P'

| P | $\vee$ | ~ | P |
|---|---|---|---|
| $\top$ | $\top$ | $\bot$ | $\top$ |
| $\iota$ | $\iota$ | $\iota$ | $\iota$ |
| $\bot$ | $\top$ | $\top$ | $\bot$ |

Excluded middle still does not come out tautologous.
But, that is a law that some folks would like to abandon, anyway.
But, is the change motivated?
The lesson of Lukasiewicz semantics is that we need not give up classical tautologies, logical truths, to have a three-valued logic.
The fewer changes we make to the set of logical truths, the less "deviant" the logic is.
But, the semantics which allows us to retain these logical truths may not be as pretty as we would like.

Lastly, consider the effect on validity of moving from Bochvar to Kleene or Lukasiewicz.
Consider again the argument: ‘P / Q ∨ P:

Bochvar:

| P | // | Q | ∨ | P |
|---|---|---|---|---|
| ⊤ | | ⊤ | ⊤ | ⊤ |
| ⊤ | | ι | ι | ⊤ |
| ⊤ | | ⊥ | ⊤ | ⊤ |
| ι | | ⊤ | ι | ι |
| ι | | ι | ι | ι |
| ι | | ⊥ | ι | ι |
| ⊥ | | ⊤ | ⊤ | ⊥ |
| ⊥ | | ι | ι | ⊥ |
| ⊥ | | ⊥ | ⊥ | ⊥ |

counter-example in Row 2

Kleene:

| P | // | Q | ∨ | P |
|---|---|---|---|---|
| ⊤ | | ⊤ | ⊤ | ⊤ |
| ⊤ | | ι | ⊤ | ⊤ |
| ⊤ | | ⊥ | ⊤ | ⊤ |
| ι | | ⊤ | ⊤ | ι |
| ι | | ι | ι | ι |
| ι | | ⊥ | ι | ι |
| ⊥ | | ⊤ | ⊤ | ⊥ |
| ⊥ | | ι | ι | ⊥ |
| ⊥ | | ⊥ | ⊥ | ⊥ |

valid - no counter-example

Lukasiewicz:

| P | // | Q | ∨ | P |
|---|---|---|---|---|
| ⊤ | | ⊤ | ⊤ | ⊤ |
| ⊤ | | ι | ⊤ | ⊤ |
| ⊤ | | ⊥ | ⊤ | ⊤ |
| ι | | ⊤ | ⊤ | ι |
| ι | | ι | ι | ι |
| ι | | ⊥ | ι | ι |
| ⊥ | | ⊤ | ⊤ | ⊥ |
| ⊥ | | ι | ι | ⊥ |
| ⊥ | | ⊥ | ⊥ | ⊥ |

valid - no counter-example

Both Kleene and Lukasiewicz semantics thus maintain some of the classical inference patterns.
See the exercises in §VI of these notes for more comparisons.

**VI. Some exercises you might try**

Construct truth tables for each of the following propositions, under classical semantics and each of the three three-valued semantics (Bochvar, Kleene, Lukasiewicz). Compare the results.

1. P ∨ ~P
2. P ⊃ P
3. (P ⊃ Q) ≡ (~P ∨ Q)
   Note: you can construct the truth table for the biconditional by remembering that ‘P≡Q’ is logically equivalent to ‘(P ⊃ Q) • (Q ⊃ P)’

Use the indirect method of truth tables to test each of the following arguments for validity, under classical semantics and each of the three three-valued semantics (B, $K_3$, and $L_3$). Compare the results.

1. P ⊃ Q
   P / Q

2. P / ~(Q • ~Q)

3. P / P ∨ Q

**VII. Problems with three-valued logics**

We have already discussed the loss of (at least some) classical tautologies and classically valid inference patterns.
Furthermore, it is not clear that all the problems that motivated three-valued logics can be solved by three-valued logics.
For example, Bochvar hoped that his semantics would solve the problems of the semantic paradoxes.
The liar sentence can be given a truth value in Bochvar semantics without paradox.
But, consider

A: A is untrue

Now, suppose A is true.
Thus, A is untrue.
But then A turns out to be true (because it says that A is untrue).
And here we go again!

Another worry about three-valued logics is that assigning a truth-value of 'unknown' involves a conceptual confusion.
'Unknown' may not be a third truth value, but merely the lack of a truth value.
Instead of filling in such cells in the truth table, we should just leave them blank.
Leaving certain cells of the truth table blank is part of what is called the truth-value gap approach.
Faced with truth-value gaps, or partial valuations, the logician may consider something called a supervaluation.
A supervaluation considers the different ways to complete partial valuations, and classifies formulas and arguments according to the possibilities for completion.
But, that is a topic for another time.

For more worries about three-valued logics, see paper topic 4, below.
In particular, Quine's worry about the deviance of three-valued logic is enormously influential.

## VIII. Avoiding three-valued logics

I introduced three-valued logics in order to respond to some problems which arose with classical logic.

M1. Mathematical sentences with unknown truth values
M2. Statements about the future
M3. Failure of presupposition
M4. Nonsense
M5. Programming needs
M6. Semantic paradoxes
M7. The paradoxes of the material conditional
M8. Vagueness

I mentioned that three-valued logics does not solve the problems of the semantic paradoxes.
There are ways for the classical logician to deal with all of these problems, anyway.
I will not discuss each of them, here, but, provide a few hints.

M1, concerning sentences with unknown truth values and M2, concerning propositions referring to future

events, are related.
In both cases, we can blame ourselves, rather than the world, for our not knowing the truth value.
Thus, we can say that Goldbach's conjecture is either true or false, but we just do not know which.
Similarly, we can say that either there will be a party at Dunham tomorrow, or there will not.
We need not ascribe a deep problem to truth values.
Such sentences have truth values.
We just do not know them.
Classical logic deals with problems about time by appealing to a four-dimensional framework.
We take a God's-eye point of view.
Going four-dimensional, we add a time-stamp to all our claims.
Then, a statement about the future is true if it ends up true at the time.
We need not see the logic as committing us to a determined future.
We just know that statements about future events will eventually have truth values.
There are also tense logics, which introduce temporal operators but maintain classical semantics, to help with time.

For failures of presupposition, M3, we can use Bertrand Russell's analysis of definite descriptions.
In the last unit of this course, we will see a more precise analysis of Russell's solution.
For now, consider again an example of failure of presupposition.

B. The woman on the moon is six feet tall.

We can analyze B to make the assumption explicit.
We can re-cast B as:

C: There is a woman on the moon and she is six feet tall.

Sentence C has the form 'P • Q'.
'P' is false, so 'P • Q' is false.
We can similarly regiment sentence D as E.

D. The woman on the moon is not six feet tall.
E: There is a woman on the moon and she is not six feet tall.

We regiment E as 'P • ~Q'
P is false, so 'P • ~Q' is false.
We thus do not have a situation in which the same proposition seems true and false.
In both cases, P is false, so the account of the falsity of both sentences D and E can be the same.
We thus lose the motivation for introducing a third truth value.

For M4, nonsense, and M6, paradoxes, and M8, vagueness, we can deny that such sentences express propositions.
We may claim that just as some strings of letters do not form words, and some strings of words do not form sentences, some grammatical sentences do not express propositions.
This would be the same as to call them meaningless.
This solution is a bit awkward, since it does seem that 'This sentence is false' is perfectly meaningful.
But if it prevents us from having to adopt three-valued logics, it might be a useful move.

**IX. Paper Topics**

1. Do assertions about the future have a truth value? Consider both the bivalent and the three-valued alternatives. You might compare Aristotle's view with that of Leibniz, who says that contingent truths are not necessary, even though they are certain. Alternatively, you could look at Haack's discussion of the way Aristotle's suggestion was pursued by Lukasiewicz. If you want to pursue an interesting technical discussion, Prior's "Three-Valued Logic and Future Contingents" is written in Polish notation.

2. How should we understand the sentence 'the king of America is not bald'? Consider Russell's theory of descriptions, and contrast it with Strawson's response. You might also consider the questions whether there a difference between logical and grammatical form, and, whether ordinary language has a logic.

3. Are there any people? Consider the problem of vagueness, and the many-valued approach to its solution.

4. Quine, in Chapter 6 of *Philosophy of Logic*, calls three-valued logic deviant, and insists that to adopt three-valued logic is to change the subject. Why does Quine prefer classical logic? Consider his maxim of minimum mutilation. Who can deal better with the problems, sketched at the beginning of these notes, that motivate three-valued logic. (You need not consider all of the problems, but you should provide a general sense of how each approach works.)

5. Compare Bochvar semantics, Kleene semantics, and Lucasiewicz semantics. What differences do the different semantics have for classical tautologies? What differences do they have for classical inferences (validity and invalidity)? Be sure to consider the semantics of the conditional. Which system seems most elegant? This paper will be mainly technical, explaining the different semantics and their results.

6. Do three-valued logics solve their motivating problems? Philosophers explore three-valued logics as a way of dealing with various problems, which I discuss in these notes. Consider some of the problems and show how one of the systems tries to resolve the problem. For this paper, I recommend, but do not insist, that you focus on Kleene's semantics. If you try to deal with Epimenides, and the semantic paradoxes, you might want to focus just on that problem.

7. Bochvar introduced a new so-called assertion operator, $\vdash$. Use of this operator allows us to recapture analogs of classical tautologies within Bochvar semantics. Describe the truth table for this operator. Show how it allows us to construct tautologies. How does the new operator affect the set of valid formulas? (It can be shown that on Bochvar semantics, any argument using only the standard operators which has consistent premises, and which contains a sentence letter in the conclusion that does not appear in any of the premises, is invalid. You might consider this result, and the effect of the new operator on it.)

**X. Suggested Readings**

Aristotle, *De Interpretatione.* In *The Complete Works of Aristotle, vol.1*, Jonathan Barnes, ed. Princeton University Press, 1984. On the sea battle, and future contingents.

Bochvar, D.A. "On a three-Valued Logical Calculus and Its Application to the Analysis of the Paradoxes of the Classical Extended Functional Calculus." *History and Philosophy of Logic* 2: 87-112,

1981.
Chomsky, Noam. *Syntactic Structures*. This book contains the discussion about colorless green ideas, but not a defense of three-valued logics. (Chomsky was arguing for a distinction between grammaticality and meaningfulness.)
Dummett, Michael. "The philosophical basis of intuitionist logic". In *Philosophy of Mathematics: Selected Readings*, 2nd ed., Paul Benacerraf and Hilary Putnam, eds. Cambridge University Press, 1983. And the selections by Heyting and Brouwer in the same volume. The intuitionists believed that an unproven mathematical statement lacked a truth value. These articles are all pretty technical, though.
Fisher, Jennifer. *On the Philosophy of Logic*. Chapters 7 and 9.
Haack, Susan. *Deviant Logic, Fuzzy Logic: Beyond the Formalism*. University of Chicago, 1996. Chapter 4 contains a discussion of Aristotle's view on future contingents, as well as more recent applications.
Haack, Susan. *Philosophy of Logics*. Cambridge, 1978. Chapters 9 and 11.
Kleene, Stephen. "On Notation for Ordinal Numbers." *The Journal of Symbolic Logic* 3.4: 150-5, 1938.
Leibniz, G.W. *Discourse on Metaphysics*. The early sections, especially §6-§13, contain his distinction between certainty and necessity.
Prior, A.N. "Three-Valued Logic and Future Contingents." *The Philosophical Quarterly* 3.13: 317-26, 1953.
Putnam, Hilary, "Three-valued logic" and "The logic of quantum mechanics", in *Mathematics, Matter and Method: Philosophical Papers, vol. 1*. Cambridge University Press, 1975. Do we need three-valued logic in order to account for oddities in quantum mechanics?
Quine, Willard van Orman. *Philosophy of Logic*, 2nd ed. Harvard University Press, 1986. The discussion of deviant logics and changing the subject is in Chapter 6, but Chapter 4, on logical truth, is exceptionally clear and fecund.
Quine, Willard van Orman. *The Ways of Paradox*. Harvard University Press, 1976. The title essay is the source of the 'yields a falsehood...' paradox, and contains an excellent discussion of paradoxes.
Read, Stephen. *Thinking about Logic*. Oxford, 1995. Chapters 6 and 7.
Russell, Bertrand. "On Denoting." In *The Philosophy of Language*, 5th ed., A.P. Martinich, ed. Oxford University Press, 2008. "On Denoting" is widely available.
Russell, Bertrand. *Introduction to Mathematical Philosophy*. Routledge, 1993. Chapter 16: "Descriptions". Contains a clearer discussion of Russell's solution to the problem of some forms of failure of presupposition than the ubiquitous "On Denoting."
Strawson, P.F. "On Referring." An alternative to Russell's theory of descriptions. Also in the Martinich collection.
Unger, Peter. "Why There Are No People." *Midwest Studies in Philosophy*, 1979.
Williamson, Timothy. *Vagueness*. Routledge, 1994. Chapter 1 has a nice discussion of the history of vagueness, and Chapter 4 discusses the three-valued logical approach to the problem.