Philosophy 240: Symbolic Logic
Fall 2010
Mondays, Wednesdays, Fridays: 9am - 9:50am

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Conditional Proof (§7.5)

I. Conditional Proof: A New Method of Derivation

When you want to derive a conditional conclusion, you can assume the antecedent of the conditional, for the purposes of the derivation, taking care to indicate the presence of that assumption later.
Procedure for conditional proof:

1. Indent, assuming the antecedent of your desired conditional.
    - Write 'ACP', for 'assumption for conditional proof'.
    - Use a vertical line to set off the assumption from the rest of your derivation.
2. Derive the consequent of desired conditional.
    - Continue the vertical line.
    - Proceed otherwise as before, using any lines already established.
3. Discharge (un-indent).
    - Write the first line of your assumption, a horseshoe, and the last line of the indented sequence.
    - Justify the un-indented line with CP, and indicate the indented line numbers.
    - Once you've discharged your assumption, you may *never* use statements within the scope of that assumption later in the proof.

So, consider:

1. $A \vee B$		
2. $B \supset (E \cdot D)$	/ $\sim A \supset D$	Note the conditional conclusion.
│ 3. $\sim A$	ACP	What if ~A were true (i.e. A were false)?
│ 4. B	1, 3, DS	
│ 5. $E \cdot D$	2, 4, MP	
│ 6. D	5, Com, Simp	Then D would be true.
7. $\sim A \supset D$	3-6, CP	So, if A were true, then D would be.

QED

Conditional proof makes many of the derivations we have done significantly easier.
Consider, from the practice sheet for the second test:

1. $(P \supset Q) \bullet (R \supset S)$	/ $(P \bullet R) \supset (Q \bullet S)$
│ 2. $P \bullet R$	ACP
│ 3. $P \supset Q$	1, Simp
│ 4. P	2, Simp
│ 5. Q	3, 4, MP
│ 6. $R \supset S$	1, Com, Simp
│ 7. R	2, Com, Simp
│ 8. S	6, 7, MP
│ 9. $Q \bullet S$	5, 8, Conj
10. $(P \bullet R) \supset (Q \bullet S)$	2-9, CP

QED

You can use CP repeatedly within the same proof, whether nested or sequentially.
This is a nested CP:

1. $P \supset (Q \vee R)$		
2. $(S \cdot P) \supset \sim Q$	/ $(S \supset P) \supset (S \supset R)$	
│ 3. $S \supset P$	ACP	Now we want $S \supset R$.
│ │ 4. S	ACP	Now we want R.
│ │ 5. P	3, 4, MP	
│ │ 6. $Q \vee R$	1, 5, MP	
│ │ 7. $S \cdot P$	4, 5, Conj	
│ │ 8. $\sim Q$	2, 7, MP	
│ │ 9. R	6, 8, DS	
│ 10. $S \supset R$	4-9, CP	
11. $(S \supset P) \supset (S \supset R)$	3-10, CP	

QED

The following derivation demonstrates how CP is useful for proving biconditionals, in sequential uses of conditional proof.

1. $(B \vee A) \supset D$		
2. $A \supset \sim D$		
3. $\sim A \supset B$ / $B \equiv D$		
│ 4. B	ACP	
│ 5. $B \vee A$	4, Add	
│ 6. D	1, 5, MP	
7. $B \supset D$	4-6 CP	
│ 8. D	ACP	Note: We can't use the B from the previous assumption.
│ 9. $\sim A$	2, 8, DN, MT	
│ 10. B	3, 9, MP	
11. $D \supset B$	8-10 CP	
12. $(B \supset D) \cdot (D \supset B)$	7, 11, Conj	
13. $B \equiv D$	12, Equiv	

QED

This should always be your first thought when proving biconditionals:

> You want: '$P \equiv Q$', which is logically equivalent to '$(P \supset Q) \cdot (Q \supset P)$'.
> Assume P, Derive Q, Discharge.
> Assume Q, Derive P, Discharge.
> Conjoin the two conditionals.
> Use Material Equivalence to yield the biconditional.

This method does not always work, but it's usually a good first thought.

You may use CP in the middle of a proof to derive statements which are not your main conclusion:

1. $P \supset (Q \cdot R)$	
2. $(P \supset R) \supset (S \cdot T)$	/ T
│3. P	ACP
│4. $Q \cdot R$	1, 3, MP
│5. R	4, Com, Simp
6. $P \supset R$	3-5, CP
7. $S \cdot T$	2, 6, MP
8. T	7, Com, Simp

QED

II. Exercises A. Derive the conclusions of each of the following arguments using the 18 rules, and the method of conditional proof.

1.
1. $A \supset B$
2. $(A \cdot B) \supset D$ / $A \supset D$

2.
1. $H \supset (E \supset F)$
2. $H \supset (G \supset F)$
3. $\sim F$ / $H \supset \sim(E \vee G)$

3.
1. $\sim L \supset M$
2. $\sim(L \cdot M)$ / $\sim M \equiv L$

4.
1. $K \supset (G \vee \sim I)$
2. $I \supset (G \supset J)$ / $K \supset (I \supset J)$

5.
1. $A \supset (B \vee D)$
2. $E \supset (\sim D \supset P)$
3. $\sim D$ / $\sim(B \vee P) \supset \sim(A \vee E)$

Solutions may vary.

III. Logical Truths

Compare the following three sentences:

A. If it is raining, then I will be unhappy.
B. If it is raining, then I will get wet.
C. If it is raining, then it is raining.

A and B are contingent sentences, each representable as ‘$P \supset Q$’
B is a little bit more compelling, but it is still possible for both sentences to be false.
C, on the other hand, can never be false, as long as we hold the meanings of the terms constant.
It is of the form ‘$P \supset P$’, and it is a logical truth, or a law of logic.

Logical truths are the theorems of our logical theory, just as certain geometric statements are theorems of Euclidean geometry.
A theory is just a set of sentences.
Theorems are the sentences that characterize a theory.
A logical system is best characterized by the set of its logical truths.

The logical truths of propositional logic are tautologies.
They do not depend on any premises.
We should be able to prove them without any premises.
Until now, though, we have had no way to construct a derivation with no premises.

Some theories, including any non-logical theory, are axiomatic.
Axiomatic logical theories normally take a few tautologies as axiom schemas.
In such a system, any sentence of the form of an axiom can be inserted into a derivation with no further justification.
Our logical theory has no axioms.
In order to produce a derivation, we have needed to take some (usually contingent) assumptions as premises.

We can use conditional proof to derive logical truths.
The following derivation shows that ‘$(P \supset Q) \supset [(Q \supset R) \supset (P \supset R)]$’ is a logical truth.

│1. $P \supset Q$	ACP
│ │2. $Q \supset R$	ACP
│ │3. $P \supset R$	1, 2, HS
│4. $(Q \supset R) \supset (P \supset R)$	2-3, CP
5. $(P \supset Q) \supset [(Q \supset R) \supset (P \supset R)]$	1-4, CP

QED

Note that the last line of the proof is further un-indented than the first line, since the first line is indented.
Be careful not to use the assigned proposition in the proof.
The conclusion is not part of the derivation until the very end.

Here is another example: Prove that ‘$[P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]$’ is a logical truth.

| 1.	$P \supset (Q \supset R)$	ACP	(to prove $(P \supset Q) \supset (P \supset R)$)
| | 2.	$P \supset Q$	ACP	(to prove $(P \supset R)$)
| | | 3.	P	ACP	(to prove R)
| | | 4.	$Q \supset R$	1, 3, MP	
| | | 5.	Q	2, 3, MP	
| | | 6.	R	4, 5, MP	
| | 7.	$P \supset R$	3-6 CP	
| 8.	$(P \supset Q) \supset (P \supset R)$	2-7, CP	
9.	$[P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]$	1-8, CP	

QED

IV. Exercises B. Derive each of the following logical truths, using CP or IP.

1. $[(A \supset B) \cdot A] \supset B$
2. $(P \vee P) \supset P$
3. $(A \supset B) \supset [A \supset (A \cdot B)]$

Solutions may vary.