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Class 39 - December 4
Derivations Using Identity II (§8.7)

I. A couple of longer proofs

We're just going to do a couple of longer proofs using identity, today.

1. (∃x)Hx		
2. (x)(y)[(Hx · Hy) ⊃ x=y]		/ (∃x)[Hx · (y)(Hy ⊃ x=y)]
3. Ha		1, EI
	| 4. ~(∃x)[Hx · (y)(Hy ⊃ x=y)]	AIP
	| 5. (x)~[Hx · (y)(Hy ⊃ x=y)]	4, CQ
	| 6. (x)[~Hx ∨ ~(y)(Hy ⊃ x=y)]	5, DM
	| 7. ~Ha ∨ ~(y)(Hy ⊃ a=y)	6, UI
	| 8. ~(y)(Hy ⊃ a=y)	7, 3, DN, DS
	| 9. (∃y) ~(Hy ⊃ a=y)	8, CQ
	| 10. ~(Hb ⊃ a=b)	9, EI
	| 11. ~(~Hb ∨ a=b)	10, Impl
	| 12. Hb · ~a=b	11, DM, DN
	| 13. Hb	12, Simp
	| 14. ~a=b	12, Com Simp
	| 15. (y)[(Ha · Hy) ⊃ a=y]	2, UI
	| 16. (Ha · Hb) ⊃ a=b	15, UI
	| 17. Ha · Hb	3, 13, Conj
	| 18. a=b	16, 17, MP
	| 19. a=b · ~a=b	18, 14, Conj
20. (∃x)[Hx · (y)(Hy ⊃ x=y)]		4-19, IP

QED

This derivation could be shortened by using the Rules of Passage, which we discussed earlier this term.

1. (∃x)Hx	
2. (x)(y)[(Hx · Hy) ⊃ x=y]	/ (∃x)[Hx · (y)(Hy ⊃ x=y)]
3. (x)(y)[Hx ⊃ (Hy ⊃ x=y)]	1, Export
4. (x)[Hx ⊃ (y)(Hy ⊃ x=y)]	RP8 (Not a rule of our system!)
5. Ha	1, EI
6. Ha ⊃ (y)(Hy ⊃ a=y)]	3, UI
7. (y)(Hy ⊃ a=y)]	6, 5, MP
8. Ha • (y)(Hy ⊃ a=y)]	5, 7, Conj
9. (∃x)[Hx · (y)(Hy ⊃ x=y)]	8, EG

QED

But, unfortunatly, we don't have the rules of passage.
So, something like the first proof is necessary.

Here is a derivation of a longer argument using ID.

> There are at least two cars in the driveway.
> All the cars in the driveway belong to John.
> John has at most two cars.
> So, there are exactly two cars in the driveway.

1. $(\exists x)(\exists y)(Cx \bullet Dx \bullet Cy \bullet Dy \bullet x \neq y)$	
2. $(x)[(Cx \bullet Dx) \supset Bxj]$	
3. $(x)(y)(z)[(Cx \bullet Bxj \bullet Cy \bullet Byj \bullet Cz \bullet Bzj) \supset (x=y \vee x=z \vee y=z)]$	
$/ (\exists x)(\exists y)\{Cx \bullet Dx \bullet Cy \bullet Dy \bullet x \neq y \bullet (z)[(Cz \bullet Dz) \supset (z=x \vee z=y)]$	
4. $(\exists y)(Ca \bullet Da \bullet Cy \bullet Dy \bullet a \neq y)$	1, EI
5. $Ca \bullet Da \bullet Cb \bullet Db \bullet a \neq b$	4, EI
6. $Ca \bullet Da$	5, Simp
7. $(Ca \bullet Da) \supset Baj$	2, UI
8. Baj	7, 6, MP
9. $Cb \bullet Db$	5, Simp
10. $(Cb \bullet Db) \supset Bbj$	2, UI
11. Bbj	10, 9, MP
	12. $\sim(z)[(Cz \bullet Dz) \supset (z=a \vee z=b)]$	AIP
	13. $(\exists z)\sim[(Cz \bullet Dz) \supset (z=a \vee z=b)]$	12, CQ
	14. $(\exists z)\sim[\sim(Cz \bullet Dz) \vee (z=a \vee z=b)]$	13, Impl
	15. $(\exists z)[(Cz \bullet Dz) \bullet \sim(z=a \vee z=b)]$	14, DM, DN
	16. $Cc \bullet Dc \bullet \sim(c=a \vee c=b)$	15, EI
	17. Ca	6, Simp
	18. $Ca \bullet Baj$	17, 8 Conj
	19. Cb	9, Simp
	20. $Cb \bullet Bbj$	19, 11, Conj
	21. $Cc \bullet Dc$	16, Simp
	22. $(Cc \bullet Dc) \supset Bcj$	2, UI
	23. Bcj	22, 21, MP
	24. Cc	21, Simp
	25. $Cc \bullet Bcj$	24, 23, Conj
	26. $Ca \bullet Baj \bullet Cb \bullet Bbj \bullet Cc \bullet Bcj$	18, 20, 25, Conj
	27. $(y)(z)[(Ca \bullet Baj \bullet Cy \bullet Byj \bullet Cz \bullet Bzj) \supset (a=y \vee x=z \vee y=z)]$	3, UI
	28. $(z)[(Ca \bullet Baj \bullet Cb \bullet Bbj \bullet Cz \bullet Bzj) \supset (a=b \vee a=z \vee b=z)]$	27, UI
	29. $(Ca \bullet Baj \bullet Cb \bullet Bbj \bullet Cc \bullet Bcj) \supset (a=b \vee a=c \vee b=c)$	28, UI
	30. $a=b \vee a=c \vee b=c$	29, 26, MP
	31. $a \neq b$	5, Simp
	32. $a=c \vee b=c$	30, 31, DS
	33. $\sim(c=a \vee c=b)$	16, Com, Simp
	34. $\sim(a=c \vee b=c)$	33, ID
	35. $(a=c \vee b=c) \bullet \sim(a=c \vee b=c)$	32, 34, Conj
36. $(z)[(Cz \bullet Dz) \supset (z=a \vee z=b)]$	12-35, IP, DN
37. $Ca \bullet Da \bullet Cb \bullet Db \bullet a \neq b \bullet (z)[(Cz \bullet Dz) \supset (z=a \vee z=b)]$	6, 9, 31, 36, Conj
38. $(\exists y)\{Ca \bullet Da \bullet Cy \bullet Dy \bullet a \neq y \bullet (z)[(Cz \bullet Dz) \supset (z=a \vee z=y)]\}$	37, EG
39. $(\exists x)(\exists y)\{Cx \bullet Dx \bullet Cy \bullet Dy \bullet x \neq y \bullet (z)[(Cz \bullet Dz) \supset (z=x \vee z=y)]\}$	38, EG

QED

II. Exercises

1.
1. $(x)[(Px \bullet Hjx) \supset Sx]$
2. $(\exists x)(Px \bullet Hjx)$
3. $(x)(y)[(Sx \bullet Hjx \bullet Sy \bullet Hjy) \supset x{=}y]$ / $(\exists x)\{Px \bullet Hjx \bullet (y)[(Py \bullet Hjy) \supset x{=}y] \bullet Sx\}$

2.
1. $(\exists x)(\exists y)(Px \bullet Rx \bullet Py \bullet Ry \bullet x \neq y)$
2. $(x)[(Px \bullet Rx) \supset Cx]$
3. $(x)(y)(z)[(Cx \bullet Cy \bullet Cz \bullet Rx \bullet Ry \bullet Rz) \supset (x{=}y \vee x{=}z \vee y{=}z)]$
/ $(\exists x)(\exists y)\{Px \bullet Rx \bullet Py \bullet Ry \bullet x \neq y \bullet (z)[(Pz \bullet Rz) \supset (z{=}x \vee z{=}y)]\}$

3.
1. $(\exists x)(\exists y)(\exists z)(Fx \bullet Fy \bullet Fz \bullet x \neq y \bullet x \neq z \bullet y \neq z)$
2. $(\exists x)\{Fx \bullet Gx \bullet (y)[(Fy \bullet Gy) \supset x{=}y]\}$
3. $(x)({\sim}Gx \supset Hx)$ / $(\exists x)(\exists y)(Hx \bullet Hy \bullet x \neq y)$

Warning! This last one may take over 75 steps!