

Class 34 - November 16
 Derivations Using Relational Predicates (§8.6)

I. Deductions using Relational Predicates and Overlapping Quantifiers

Consider the argument with which we started studying relations:

Bob is taller than Charles.
 Andrew is taller than Bob.
 For any x , y and z , If x is taller than y and y is taller than z , then x is taller than z .
 So, Andrew is taller than Charles.

1. Tbc
2. Tab
3. $(x)(y)(z)[(Txy \cdot Tyz) \supset Txz]$ / Tac

To derive the conclusion, we use the same rules of inference we used in monadic predicate logic.

Remove quantifiers one at a time.

Take care to make appropriate instantiations, to variables or constants.

There is only one exception, to UG, which we will note shortly.

1. Tbc
2. Tab
3. $(x)(y)(z)[(Txy \cdot Tyz) \supset Txz]$ / Tac
4. $(y)(z)[(Tay \cdot Tyz) \supset Taz]$ 3, UI
5. $(z)[(Tab \cdot Tbz) \supset Taz]$ 4, UI
6. $(Tab \cdot Tbc) \supset Tac$ 5, UI
7. $(Tab \cdot Tbc)$ 2, 1, Conj
8. Tac 6, 7, MP

QED

Here is another example, in which we take off the quantifiers in the middle of the proof, rather than at the beginning.

1. $(\exists x)[Hx \cdot (y)(Hy \supset Lyx)]$ / $(\exists x)(Hx \cdot Lxx)$
2. $Ha \cdot (y)(Hy \supset Lya)$ 1, EI
3. Ha 2, Simp
4. $(y)(Hy \supset Lya)$ 2, Com, Simp
5. $Ha \supset Laa$ 4, UI
6. Laa 5, 3, MP
7. $Ha \cdot Laa$ 3, 6, Conj
8. $(\exists x)(Hx \cdot Lxx)$ 7, EG

QED

II. The Restriction on UG:

Consider the following derivation, beginning with a proposition that can be interpreted as ‘Everything loves something’.

- | | | |
|------------------------|-------|-------------------------------|
| 1. $(x)(\exists y)Lxy$ | | |
| 2. $(\exists y)Lxy$ | 1, UI | |
| 3. Lxa | 2, EI | |
| 4. $(x)Lxa$ | 3, UG | Note: this step is incorrect! |
| 5. $(\exists y)(x)Lxy$ | 4, EG | |

Given our interpretation of line 1, line 5 reads, ‘There’s something that everything loves’.

It does not follow from the proposition that everything loves something that there is one thing that everything loves.

Imagine we ordered all the things in a circle, and everyone loved just the thing to its left.

Line 1 would be true, but line 5 would be false.

So, we should not be able to derive step 5 from step 1.

We can locate the problem in step 4 of the above derivation.

In line 2 we universally instantiated to some random object x .

So ‘ x ’ could have stood for any object.

It retained its universal character, even without a universal quantifier to bind it.

But, in line 3, when we existentially instantiated, we gave a name, ‘ a ’ to the thing which bore relation L to it, to the thing x loves.

Once we gave a name to the thing that x loves, x lost its universal character.

It could no longer be anything, which loves something.

It now had to be the thing that loves a .

‘ x ’ became as particular an object as ‘ a ’ is.

So, the generalization at line 4 must be blocked.

Variables lose their universal character if they are free when EI is used.

So, you may never UG on a variable when there’s a constant present, and the variable was free when the constant was introduced.

I.e. We can not UG In line 4, because ‘ x ’ was free in line 3 when ‘ a ’ was introduced.

Here is a derivation with an acceptable use of UG:

- | | |
|---|-----------------------|
| 1. $(\exists x)(y)[(\exists z)Ayz \supset Ayx]$ | |
| 2. $(y)(\exists z)Ayz$ | / $(\exists x)(y)Ayx$ |
| 3. $(y)[(\exists z)Ayz \supset Aya]$ | 1, EI |
| 4. $(\exists z)Ayz \supset Aya$ | 3, UI |
| 5. $(\exists z)Ayz$ | 2, UI |
| 6. Aya | 4, 5, MP |
| 7. $(y)Aya$ | 6, UG |
| 8. $(\exists x)(y)Ayx$ | 7, EG |

QED

Note that at line 7, UG is acceptable because ‘ y ’ was not free when ‘ a ’ was introduced in line 3!

This restriction only applies to UG; all other rules are just as they are in monadic predicate logic.

III. More examples, and a warning about accidental binding

When quantifying, i.e. when using UG or EG, watch for accidental binding.

Consider :

$$(Pa \cdot Qa) \supset (Fx \vee Gx)$$

If you try to quantify over the ‘a’ using EG with the variable ‘x’, you accidentally bind the latter two terms:

$$(\exists x)[(Px \cdot Qx) \supset (Fx \vee Gx)]$$

That is a legitimate use of existential generalization, but it may not mean what you want.

Instead, use a ‘y’:

$$(\exists y)[(Py \cdot Qy) \supset (Fx \vee Gx)]$$

Now the latter terms remain free.

We can still bind them with either a universal quantifier or an existential quantifier, later.

$$\begin{aligned} & (x)(\exists y)[(Py \cdot Qy) \supset (Fx \vee Gx)] \\ & (\exists x)(\exists y)[(Py \cdot Qy) \supset (Fx \vee Gx)] \end{aligned}$$

Here is a conditional proof using relational predicates:

1. $(x)[Ax \supset (y)Bxy]$	
2. $(x)[Ax \supset (\exists y)Dyx]$	$\neg(x)[Ax \supset (\exists y)(Bxy \cdot Dyx)]$
3. Ax	ACP
4. $Ax \supset (y)By$	1, UI
5. $Ax \supset (\exists y)Dyx$	2, UI
6. $(y)Bxy$	4, 3, MP
7. $(\exists y)Dyx$	5, 3 MP
8. Dax	7, EI
9. Bxa	6, UI
10. $Bxa \cdot Dax$	9, 8, Conj
11. $(\exists y)(Bxy \cdot Dyx)$	10, EG
12. $Ax \supset (\exists y)(Bxy \cdot Dyx)$	3-11, CP
13. $(x)[Ax \supset (\exists y)(Bxy \cdot Dyx)]$	12, UG

QED

Here is a more complex proof:

1. $(x)(Wx \supset Xx)$	
2. $(x)[(Yx \cdot Xx) \supset Zx]$	
3. $(x)(\exists y)(Yy \cdot Ayx)$	
4. $(x)(y)[(Ayx \cdot Zy) \supset Zx]$	$/ (x)[(y)(Ayx \supset Wy) \supset Zx]$
5. $(y)(Ayx \supset Wy)$	ACP
6. $(\exists y)(Yy \cdot Ayx)$	3, UI
7. $Ya \cdot Aax$	6, EI
8. $Aax \supset Wa$	5, UI
9. Aax	7, Com, Simp
10. Wa	8,9, MP
11. $Wa \supset Xa$	1, UI
12. Xa	11, 10, MP
13. Ya	7, Simp
14. $Ya \cdot Xa$	13, 12, Conj
15. $(Ya \cdot Xa) \supset Za$	2, UI
16. Za	15, 14, MP
17. $(y)[(Ayx \cdot Zy) \supset Zx]$	4, UI
18. $(Aax \cdot Za) \supset Zx$	17, UI
19. $Aax \cdot Za$	9, 16, Conj
20. Zx	18, 19, MP
21. $(y)(Ayx \supset Wy) \supset Zx$	5-20, CP
22. $(x)[(y)(Ayx \supset Wy) \supset Zx]$	21, UG

QED

Notes:

At line 16, you might be tempted to discharge your assumption and finish your CP.

But, you wouldn't be able to UG over the 'Za'.

We have to UI at line 17, retaining a variable for the predicate 'Z'.

IV. Exercises. Derive the conclusions of each of the following arguments.

1. $(x)(Cax \supset Dxb)$
 2. $(\exists x)Dxb \supset (\exists y)Dby$ $/ (\exists x)Cax \supset (\exists y)Dby$
1. $(x)[Ex \supset (y)(Fy \supset Gxy)]$
 2. $(\exists x)[Ex \cdot (\exists y)\sim Gxy]$ $/ (\exists x)\sim Fx$
1. $(\exists x)Ax \supset (\exists x)Bx$ $/ (\exists y)(x)(Ax \supset By)$
1. $(x)[Mx \supset (y)(Ny \supset Oxy)]$
 2. $(x)[Px \supset (y)(Oxy \supset Qy)]$ $/ (\exists x)(Mx \cdot Px) \supset (y)(Ny \supset Qy)$