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Translation Using Relational Predicates II (§8.6)

I. Quantifiers: narrow and wide scope

When you have multiple quantifiers in a proposition, they can either take wide scope, by standing in front of the proposition, or take narrow scope, by being located inside the proposition.
I have advised you, when translating, only to introduce quantifiers when needed (i.e. give them narrow scope).
Mostly, it is good form to keep narrow scope.
On occasion, we will just put all quantifiers in front, using wide scope.
But, moving quantifiers around is not always simple, and we must be careful.

In some cases, we can move quantifiers around without much worry.
For example, if all quantifiers are universal, we can pull them in or out at will, as long as we are careful not to accidentally bind any variables.
'Everyone loves everyone' can be written as any of the following:

1. $(x)[Px \supset (y)(Py \supset Lxy)]$
2. $(x)(y)[(Px \cdot Py) \supset Lxy]$
3. $(y)(x)[(Px \cdot Py) \supset Lxy]$

Technically, 3 is 'everyone is loved by everyone'.
But all three statements are logically equivalent.
Similarly, 'someone loves someone' can be written as any of the following:

4. $(\exists x)[Px \cdot (\exists y)(Py \cdot Lxy)]$
5. $(\exists x)(\exists y)[(Px \cdot Py) \cdot Lxy]$
6. $(\exists y)(\exists x)[(Px \cdot Py) \cdot Lxy]$

6 is 'someone is loved by someone'.
Again, 4-6 are all equivalent.

When you mix universal quantifiers with existential quantifiers, you have to be careful, since reversing the order of the quantifiers changes the meaning of the proposition.
None of the following examples are equivalent:

7. Everyone loves someone:	$(x)(\exists y)[Px \supset (Py \cdot Lxy)]$
8. Everyone is loved by someone:	$(x)(\exists y)[Px \supset (Py \cdot Lyx)]$
9. Someone loves everyone:	$(\exists x)(y)[Px \cdot (Py \supset Lxy)]$
10. Someone is loved by everyone:	$(\exists x)(y)[Px \cdot (Py \supset Lyx)]$

Note that the first word in each translation above corresponds to the leading quantifier.
Also, note that the connectives which directly follow the 'Px' and the 'Py' are determined by the quantifier binding that variable.

This tendency is clearer if we take the quantifiers inside.

7'. (x)[Px ⊃ (∃y)(Py · Lxy)]
8'. (x)[Px ⊃ (∃y)(Py • Lyx)]
9'. (∃x)[Px · (y)(Py ⊃ Lxy)]
10'. (∃x)[Px · (y)(Py ⊃ Lyx)]

While all of those shifts of quantifiers are acceptable, moving quantifiers within a proposition is tricky. For example, the following sentences are *not* equivalent

11. (x)[(∃y)Lxy ⊃ Hx]
12. (x)(∃y)(Lxy ⊃ Hx)

From 11 to 12, we have moved the existential quantifier out front, and merely brought the 'Hx' into the scope of '(∃y)', which does not bind it.
11 can be interpreted as 'All lovers are happy', or 'for any x, if there is a y that x loves, then x is happy'.
In that case, 12 would be 'For any x, there is a y such that if x loves y then x is happy'.
11 does not commit to the existence of something that, by being loved, makes a person happy.
12 does.
Consider the universe in which there are things that can never be happy, for which nothing could make them happy.
11 could still be true, but 12 would have to be false.
We need a set of rules to determine which moves of quantifiers are acceptable.

II. Rules of passage

There are metalogical proofs which require that every statement of **F** to be equivalent to a statement with all quantifiers having wide scope.
Such a form is called prenex normal form (PNF).
We have already seen disjunctive normal form, in propositional logic, when we did the proofs of adequate sets of connectives.
In order to transform formulas to PNF, we can use what are sometimes called rules of passage, but which are really just rules of replacement.[1]
All ten of the following rules are taken from W.V. Quine, *Methods of Logic*, Harvard University Press, 1982; though they do not appear in exactly the form that follows.
These rules do not appear in Hurley, and I will not require that you use them in proofs.
But, they may be useful in learning how to translate.

In all ten transformation rules, 'α' stands for any formula which does not contain a free instance of the quantifier variable. (So, there's no accidental binding, or accidental removing from binding.)

[1] Quine notes that the rules of passage were so-called by Herbrand, in 1930, but were present in Whitehead and Russell's *Principia Mathematica*. Prenex normal form was used by Skolem for his proof procedure, in 1922.

Rules of passage

RP1:	$(\exists x)(Fx \vee Gx)$	::	$(\exists x)Fx \vee (\exists x)Gx$
RP2:	$(x)(Fx \bullet Gx)$	::	$(x)Fx \bullet (x)Gx$
RP3:	$(\exists x)(\alpha \bullet Fx)$	::	$\alpha \bullet (\exists x)Fx$
RP4:	$(x)(\alpha \bullet Fx)$	::	$\alpha \bullet (x)Fx$
RP5:	$(\exists x)(\alpha \vee Fx)$	::	$\alpha \vee (\exists x)Fx$
RP6:	$(x)(\alpha \vee Fx)$	::	$\alpha \vee (x)Fx$
RP7:	$(\exists x)(\alpha \supset Fx)$	::	$\alpha \supset (\exists x)Fx$
RP8:	$(x)(\alpha \supset Fx)$	::	$\alpha \supset (x)Fx$
RP9:	$(\exists x)(Fx \supset \alpha)$	::	$(x)Fx \supset \alpha$
RP10:	$(x)(Fx \supset \alpha)$	::	$(\exists x)Fx \supset \alpha$

Let's look at a few examples.
13 and 14 are equivalent by RP4.

13. $(\exists x)[Px \bullet (y)(Qy \supset Rxy)]$
14. $(\exists x)(y)[Px \bullet (Qy \supset Rxy)]$

If we didn't have RP4, we could show their equivalence by deriving 13 from 14 and 14 from 13.
'$\alpha \vdash \beta$' means that β can be derived from α; '$\vdash$' is the meta-linguistic form of '$\supset$'
Its negation is normally written with a slash through it, but I don't have easy access to that symbol, so I will write '$\sim\vdash$'.
(We haven't discussed how the rules of inference have to be restricted when using relational predicates, but the change is small and all of the derivations in these notes are acceptable.)

$13 \vdash 14$

1. $(\exists x)[Px \bullet (y)(Qy \supset Rxy)]$	
2. $Pa \bullet (y)(Qy \supset Ray)$	1, EI
	3. Qy	ACP
	4. $(y)(Qy \supset Ray)$	2, Com, Simp
	5. $Qy \supset Ray$	4, UI
	6. Ray	5, 3, MP
7. $Qy \supset Ray$	3-6, CP
8. Pa	2, Simp
9. $Pa \bullet (Qy \supset Ray)$	8, 7, Conj
10. $(y)[Pa \bullet (Qy \supset Ray)]$	9, UG
11. $(\exists x)(y)[Px \bullet (Qy \supset Rxy)]$	10, EG

QED

14 $\vdash$ 13

1. $(\exists x)(y)[Px \bullet (Qy \supset Rxy)]$	
2. $(y)[Pa \bullet (Qy \supset Ray)]$	1, EI
3. $Pa \bullet (Qy \supset Ray)$	2, UI
4. $Qy \supset Ray$	3, Com, Simp
5. $(y)(Qy \supset Ray)$	4, UG
6. Pa	3, Simp
7. $Pa \bullet (y)(Qy \supset Ray)$	6, 5, Conj
8. $(\exists x)[Px \bullet (y)(Qy \supset Rxy)$	7, EG

QED

15 and 16 are equivalent by RP8:

15. $(\exists x)(y)[Px \supset (Qy \supset Rxy)]$
16. $(\exists x)[Px \supset (y)(Qy \supset Rxy)]$

12, above, is equivalent to 17 by RP9.

17. $(x)[(y)Lxy \supset Hx]$

That transformation might strike one as strange.
It might even make one call RP9 into question.
But, notice the following:

12. $(x)(\exists y)(Lxy \supset Hx)$		
18. $(x)(\exists y)({\sim}Lxy \vee Hx)$	12, Impl	
19. $(x)(\exists y)(Hx \vee {\sim}Lxy)$	18, Com	
20. $(x)[Hx \vee (\exists y){\sim}Lxy]$	19, RP5	
21. $(x)[(\exists y){\sim}Lxy \vee Hx]$	20, Com	
22. $(x)[{\sim}(y)Lxy \vee Hx]$	21, CQ	
23. $(x)[(y)Lxy \supset Hx]$	22, Impl	(Note that 23 is the same as 17.)

11, above, is equivalent, by RP10, to

24. $(x)(y)(Lxy \supset Hx)$

That should feel right, since both 11 and 24 can be interpreted as, "If anyone loves someone, then s/he is happy."

25 and 26 are equivalent, also by RP10.

25. $(x)[Px \supset (\exists y)Qy]$
26. $(\exists x)Px \supset (\exists y)Qy$

III. Proving the equivalence of RP10

We will not prove the equivalence of all of the Rules of Passage.
Most of them are quite intuitive.
RP9 and RP10 are the two oddballs.
Let's take a moment to prove RP10.

Consider first what happens when α is true, and then when α is false.
(As an example, in 25, α is '(∃x)Qy)'.)

If α is true, then both formulas will turn out to be true.

The consequent of the formula on the right is just α.

So, if α is true, the whole formula on the right will be true.

'Fx ⊃ α' will be true for every instance of x, since the consequent is true.

So, the universal generalization of each such formula (which is the formula on the left) will be true.

If α is false, then the truth value of each formula will depend.

To show that the truth values of each formula will be the same, we will show that the formula on the right is true in every case that the formula on the left is true and that the formula on the left is true in every case that the formula on the right is true.

If the formula on the left turns out to be true when α is false, it must be because 'Fx' is false, for every x.

But then, '(∃x)Fx' will be false, and so the formula on the right turns out to be true.

If the formula on the right turns out to be true, then it must be because '(∃x)Fx' is false.

And so, there will be no value of 'x' that makes 'Fx' true, and so the formula on the right will also turn out to be (vacuously) true.

IV. Using the Rules of Passage in Translations

RP10 allows us to translate 'If anything was damaged, then everyone gets upset' in two ways:

27. (∃x)Dx ⊃ (x)(Px ⊃ Ux)
28. (x)[Dx ⊃ (y)(Py ⊃ Uy)]

That is, 27 is logically equivalent to 28.

Using the RPs, we can transform any statement of predicate logic into prenex normal form, with all the quantifiers out front.
Consider the solution to §8.6: I.24, "If there are any cheaters, then if all referees are vigilant, they will be punished."

29. (x){Cx ⊃ [(y)(Ry ⊃ Vy) ⊃ Px]}
29'. (x){Cx ⊃ (∃y)[(Ry ⊃ Vy) ⊃ Px]}   RP9
29''. (x)(∃y){Cx ⊃ [(Ry ⊃ Vy) ⊃ Px]}   RP7

Of course, it would be unlikely that any one would translate the sentence as either of these equivalents.

V. Prenex Normal Form

An interesting fact about prenex normal form is that it is not the case that a given formula has a unique prenex form.
For example, consider this sentence from Quine

30: If there is a philosopher whom all philosophers contradict, then there is a philosopher who contradicts himself.

31 (∃x)[Fx • (y)(Fy ⊃ Gyx)] ⊃ (∃x)(Fx • Gxx)

In order to put this sentence into prenex form, we have first to change the ‘x’s to ‘z’s, so that when we stack the quantifiers in front, we won’t get accidental binding.

32: (∃x)[Fx • (y)(Fy ⊃ Gyx)] ⊃ (∃z)(Fz • Gzz)

In the first set of transformations to prenex form, I will work with the ‘z’, then the ‘y’.

	(∃z)(∃x){[Fx • (y)(Fy ⊃ Gyx)] ⊃ (Fz • Gzz)}	by RP7
	(∃z)(∃x){(y)[Fx • (Fy ⊃ Gyx)] ⊃ (Fz • Gzz)}	by RP4
33:	(∃z)(∃x)(∃y){[Fx • (Fy ⊃ Gyx)] ⊃ (Fz • Gzz)}	by RP9

In the second set, I will work with the ‘x’, then the ‘y’, then the ‘z’.

	(x){[Fx • (y)(Fy ⊃ Gyx)] ⊃ (∃z)(Fz • Gzz)}	by RP10
	(x){(y)[Fx • (Fy ⊃ Gyx)] ⊃ (∃z)(Fz • Gzz)}	by RP4
	(x)(∃y){[Fx • (Fy ⊃ Gyx)] ⊃ (∃z)(Fz • Gzz)}	by RP9
34:	(x)(∃y)(∃z){[Fx • (Fy ⊃ Gyx)] ⊃ (Fz • Gzz)}	by RP7

33 and 34 are equivalent to 32 (and 31).
33 and 34 are both in prenex form.
But, they differ in form from each other.

There are (I think) two other prenex forms equivalent to 32.
See if you can work them out.

VI. More entailments, and some non-entailments

Let's look at some more entailments and equivalences in quantificational logic, and some statements that are not equivalent.
35 ⊢ 36, but 36 ~⊢ 35.

35. (∃x)[Px • (y)(Qy ⊃ Rxy)]
36. (∃x)(y)[Px ⊃ (Qy ⊃ Rxy)]

35 ⊢ 36

1. (∃x)[Px • (y)(Qy ⊃ Rxy)]	
	2. ~(∃x)(y)[Px ⊃ (Qy ⊃ Rxy)]	AIP
	3. (x)(∃y)~[Px ⊃ (Qy ⊃ Rxy)]	2, CQ
	4. (x)(∃y)~[~Px ∨ ~Qy ∨ Rxy]	3, Impl, Impl
	5. (x)(∃y)(Px • Qy • ~Rxy)	4, DM, DN
	6. Pa • (y)(Qy ⊃ Ray)	1, EI
	7. (∃y)(Pa • Qy • ~Ray)	5, UI
	8. Pa • Qb • ~Rab	7, EI
	9. (y)(Qy ⊃ Ray)	6, Com, Simp
	10. Qb ⊃ Rab	9, UI
	11. Qb	8, Com, Simp
	12. Rab	10, 11, MP
	13. ~Rab	8, Com, Simp
	14. Rab • ~Rab	12, 13, Conj
15. (∃x)(y)[Px ⊃ (Qy ⊃ Rxy)]	2-14, IP, DN

QED

To see that 36 ~⊢ 35, we can construct a counter-example in a universe with two-members
I'll expand 36 in two steps, first removing the existential quantifier, then the universal.

36'. (y)[Pa ⊃ (Qy ⊃ Ray)] ∨ (y)[Pb ⊃ (Qy ⊃ Rby)]
36". {[Pa ⊃ (Qa ⊃ Raa)] • [Pa ⊃ (Qb ⊃ Rab)]} ∨ {[Pb ⊃ (Qa ⊃ Rba)] • [Pb ⊃ (Qb ⊃ Rbb)]}

I'll do the same for 35:

35'. [Pa • (y)(Qy ⊃ Ray)] ∨ [Pb • (y)(Qy ⊃ Rby)]
35". [Pa • (Qa ⊃ Raa) • (Qb ⊃ Rab)] ∨ [Pb • (Qa ⊃ Rba) • (Qb ⊃ Rbb)]

To form the counter-example, just assign false to both 'Pa' and 'Pb'.
Then, both conjuncts in 35" are false, but all the conditionals in 36" are (vacuously) true.

Here are some more entailments, and non-entailments, in metalogical form.
You could demonstrate the entailments by considering a specific instance of each.
You could prove the non-entailments by instantiating each one and constructing a counter-example, as I did just above.

37. $(x)\mathcal{F}x \lor (x)\mathcal{G}x \;\vdash\; (x)(\mathcal{F}x \lor \mathcal{G}x)$

But

38. $(x)(\mathcal{F}x \lor \mathcal{G}x) \;\sim\vdash\; (x)\mathcal{F}x \lor (x)\mathcal{G}x$

To see 38, just substitute 'P' for $\mathcal{F}$ and '~P' for $\mathcal{G}$

39. $(\exists x)(\mathcal{F}x \bullet \mathcal{G}x) \;\vdash\; (\exists x)\mathcal{F}x \bullet (\exists x)\mathcal{G}x$

But

40. $(\exists x)\mathcal{F}x \bullet (\exists x)\mathcal{G}x \;\sim\vdash\; (\exists x)(\mathcal{F}x \bullet \mathcal{G}x)$

41. $(x)(\mathcal{F}x \supset \alpha) \;\vdash\; (x)\mathcal{F}x \supset \alpha$

But

42. $(x)\mathcal{F}x \supset \alpha \;\sim\vdash\; (x)(\mathcal{F}x \supset \alpha)$

43. $(\exists x)\mathcal{F}x \supset \alpha \;\vdash\; (\exists x)(\mathcal{F}x \supset \alpha)$
e.g. $(\exists x)Px \supset (\exists y)Qy \;\vdash\; (\exists x)[Px \supset (\exists y)Qy]$

But

44. $(\exists x)(\mathcal{F}x \supset \alpha) \;\sim\vdash\; (\exists x)\mathcal{F}x \supset \alpha$
e.g. $(\exists x)[Px \supset (\exists y)Qy] \;\sim\vdash\; (\exists x)Px \supset (\exists y)Qy$

VII. Logical truths

Here are four logical truths of **F**.

45. $(y)[(x)Fx \supset Fy]$
46. $(y)[Fy \supset (\exists x)Fx]$
47. $(\exists y)[Fy \supset (x)Fx]$
48. $(\exists y)[(\exists x)Fx \supset Fy]$

These are all provable using IP.
Note that each one has a similarity to one of the four rules for removing or replacing quantifiers.

VIII. For further work

Notice that the rules of passage do not include transformations for the biconditional.
If you want something sort of fun to do, see if you can determine the relations among 49-52.

49. $(\exists x)(\alpha \equiv Fx)$
50. $\alpha \equiv (\exists x)Fx$
51. $(x)(\alpha \equiv Fx)$
52. $\alpha \equiv (x)Fx$

IX. Exercises. Translate each of the following sentences into predicate logic.

1. Everyone loves something. (Px, Lxy)
2. No one knows everything. (Px, Kxy)
3. No one knows everyone.
4. Every woman is stronger than some man. (Wx, Mx, Sxy: x is stronger than y)
5. No cat is smarter than any horse. (Cx, Hx, Sxy: x is smarter than y)
6. Dead men tell no tales. (Dx, Mx, Tx, Txy: x tells y)
7. There is a city between New York and Washington. (Cx, Bxyz: y is between x and z)
8. Everyone gives something to someone. (Px, Gxyz: y gives x to z)
9. A dead lion is more dangerous than a live dog. (Ax: x is alive, Lx, Dx, Dxy: x is more dangerous than y)
10. A lawyer who pleads his own case has a fool for a client. (Lx, Fx, Pxy: x pleads y's case; Cxy: y is a client of x)

X. Solutions

1. $(x)[Px \supset (\exists y)Lxy]$
2. $(x)[Px \supset (\exists y){\sim}Kxy]$ or $\sim(\exists x)[Px \cdot (y)Kxy]$
3. $(x)[Px \supset (\exists y)(Py \cdot {\sim}Kxy)]$ or $\sim(\exists x)[Px \cdot (y)(Py \supset Kxy]$
4. $(x)[Wx \supset (\exists y)(My \cdot Sxy)]$
5. $\sim(\exists x)[Cx \cdot (\exists y)(Hy \cdot Sxy)]$ or $(x)[Cx \supset (y)(Hy \supset {\sim}Sxy)]$
6. $(x)[(Dx \cdot Mx) \supset \sim(\exists y)(Ty \cdot Txy)]$
7. $(\exists x)(Cx \cdot Bnxw)$
8. $(x)[Px \supset (\exists y)(\exists z)(Pz \cdot Gyxz)]$
9. $(x)\{({\sim}Ax \cdot Lx) \supset (y)[(Ay \cdot Dy) \supset Dxy]\}$
10. $(x)[(Lx \cdot Pxx) \supset (\exists y)(Fy \cdot Cxy)]$ or $(x)[(Lx \cdot Pxx) \supset Fx]$
    Note that these two translations for 10 are not equivalent.
    The first translates the surface grammar.
    The second translates the meaning.