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Indirect Proof (§7.6)

**I. Indirect Proof: Another Method for Derivation**

Consider the following two proofs:

| | | |
|---|---|---|
| 1. | 1. A · ~A | / B |
| | 2. A | 1, Simp |
| | 3. A ∨ B | 2, Add |
| | 4. ~A | 1, Com, Simp |
| | 5. B | 3, 4, DS |

QED

| | | |
|---|---|---|
| 2. | 1. B ⊃ (P · ~P) | / ~B |
| | │2. B | |
| | │3. P · ~P | |
| | │4. P | |
| | │5. P ∨ ~B | |
| | │6. ~P | |
| | │7. ~B | |
| | 8. B ⊃ ~B | |
| | 9. ~B ∨ ~B | |
| | 10. ~B | |

QED

The moral of the first is that anything follows from a contradiction.
The moral of the second is that if a statement entails a contradiction, then its negation is true.
Indirect proof is based on these two morals.
Indirect proof is also called reductio ad absurdum, or just reductio.
Assume your desired conclusion is false, and try to get a contradiction.
If you get it, then you know the opposite of the assumption is true.

*Procedure for Indirect Proof, (IP)*

1. Indent, assuming the opposite of what you want to conclude (one more or one fewer '~').
2. Derive a contradiction, using any letter.
3. Discharge the negation (not the opposite!) of your assumption.

A contradiction is any statement of the form: α • ~α
The following wffs are all contradictions:

P •~ P
~~P • ~~~P
~(P∨ ~Q) • ~~(P ∨ ~Q)

Sample Derivation:

| | | |
|---|---|---|
| 1. A ⊃ B | | |
| 2. A ⊃ ~B | /~A | |
| \| 3. A | AIP | Let's see what happens if the opposite of the conclusion is true. |
| \| 4. B | 1, 3, MP | |
| \| 5. ~B | 2, 3, MP | |
| \| 6. B · ~B | 4, 5, Conj | This is impossible - a contradiction. |
| 7. ~A | 3-6, IP | So ~~A must be false, and so ~A is true. |

QED

The method of indirect proof is especially useful for proving disjunctions as well as simple statements and negations.

**II. More sample derivations**

Plain indirect proof:

| | |
|---|---|
| 1. F ⊃ ~D | |
| 2. D | |
| 3. (D · ~E) ⊃ F | / E |
| \| 4. ~E | AIP |
| \| 5. D · ~E | 2, 4, Conj |
| \| 6. F | 3, 5, MP |
| \| 7. ~D | 1, 6, MP |
| \| 8. D · ~D | 2, 7, Conj |
| 9. ~~E | 4-8, CP |
| 10. E | 9, DN |

QED

Indirect proof with conditional proof:

| | | |
|---|---|---|
| 1. E ⊃ (A · D) | | |
| 2. B ⊃ E | / (E ∨ B) ⊃ A | |
| \| 3. E ∨ B | | ACP |
| \| \| 4. ~A | | AIP |
| \| \| 5. ~A ∨ ~D | | 4, Add |
| \| \| 6. ~(A · D) | | 5, DM |
| \| \| 7. ~E | | 1, 6, MT |
| \| \| 8. B | | 3, 7, DS |
| \| \| 9. ~B | | 2, 7, MT |
| \| \| 10. B · ~B | | 8, 9, Conj |
| \| 11. ~~A | | 4-10, IP |
| \| 12. A | | 11, DN |
| 12. (E ∨ B) ⊃ A | | 3-12, CP |

QED

This last method has the form of many (almost all?) proofs in mathematics.
First, one states one's assumptions, one's specific axioms.
Then one assumes one's conclusion is false, and derives a contradiction.

**III. Exercises A**. Derive the conclusions of the following arguments using the 18 rules, and either CP or IP.

1.
  1. A ⊃ B
  2. ~A ∨ ~B / ~A

2.
  1. F ⊃ (~E ∨ D)
  2. F ⊃ ~D / F ⊃ ~E

3.
  1. ~J ⊃ (G · H)
  2. G ⊃ I
  3. H ⊃ ~I / J

4.
  1. S ⊃ (T ∨ U)
  2. W ⊃ ~U / S ⊃ ~(W · ~T)

5.
  1. (L ⊃ M) · (N ⊃ O)
  2. (M ∨ O) ⊃ P
  3. ~P / ~(L ∨ N)

**IV. Using IP to derive logical truths**

Like conditional proof, the method of indirect proof is easily adapted to proving logical truths.

To prove that '~[(X ≡ Y) · ~(X ∨ ~Y)]' is a logical truth, we start with an assumption.

| | |
|---|---|
| \| 1. (X ≡ Y) · ~(X ∨ ~Y) | AIP |
| \| 2. X ≡ Y | 1, Simp |
| \| 3. (X ⊃ Y) · (Y ⊃ X) | 2, Equiv |
| \| 4. ~(X ∨ ~Y) | 1, Com, Simp |
| \| 5. ~X · Y | 4, DM DN |
| \| 6. Y ⊃ X | 3, Com, Simp |
| \| 7. ~X | 5, Simp |
| \| 8. ~Y | 6, 7, MT |
| \| 9. Y | 5, Com, Simp |
| \| 10. Y · ~Y | 9, 8, Conj |
| 11. ~[(X ≡ Y) · ~(X ∨ ~Y)] | 1-10, IP |

QED

Here is another example: Show that ‘(P ⊃ Q) ∨ (~Q ⊃ P)’ is a logical truth.

| | |
|---|---|
| │ 1. ~[(P ⊃ Q) ∨ (~Q ⊃ P)] | AIP |
| │ 2. ~(P ⊃ Q) · ~(~Q ⊃ P) | 1, DM |
| │ 3. ~(P ⊃ Q) | 2, Simp |
| │ 4. ~(~P ∨ Q) | 3, Impl |
| │ 5. P · ~Q | 4, DM, DN |
| │ 6. ~(~Q ⊃ P) | 2, Com, Simp |
| │ 7. ~(Q ∨ P) | 6, Impl, DN |
| │ 8. ~Q · ~P | 7, DM |
| │ 9. P | 5, Simp |
| │ 10. ~P | 8, Com, Simp |
| │ 11. P · ~P | 9, 10, Conj |
| 12. (P ⊃ Q) ∨ (~Q ⊃ P) | 1-11, IP |

QED

Here are some hints to help determine whether to use conditional proof or indirect proof to derive a logical truth.

If the main connective is a conditional or a biconditional, we generally use conditional proof.
If the main connective is a disjunction or a negation, we generally use indirect proof.
If the main connective is a conjunction, we look to the main connectives of each conjunct to determine the best method of proof.

Sometimes, we might need a logical truth as an intermediate step in a proof:

| | |
|---|---|
| 1. B ⊃ [(D ⊃ D) ⊃ E] | |
| 2. E ⊃ {[F ⊃ (G ⊃ F)] ⊃ (H · ~H)} | / ~B |
| │ 3. B | AIP |
| │ 4. (D ⊃ D) ⊃ E | 1, 3, MP |
| │ │ 5. ~(D ⊃ D) | AIP |
| │ │ 6. ~(~D ∨ D) | 5, Impl |
| │ │ 7. D · ~D | 6, DM, DN |
| │ 8. D ⊃ D | 5-7, IP, DN |
| │ 9. E | 4, 8, MP |
| │ 10. [F ⊃ (G ⊃ F)] ⊃ (H · ~H) | 2, 9, MP |
| │ │ 11. F | ACP |
| │ │ 12. F ∨ ~G | 11, Add |
| │ │ 13. ~G ∨ F | 12, Com |
| │ │ 14. G ⊃ F | 13, Impl |
| │ 15. F ⊃ (G ⊃ F) | 11-14, CP |
| │ 16. H · ~H | 10, 15, MP |
| 17. ~B | 3-16, IP |

QED

Here are two observation about the above proof.

At step 4, ‘D ⊃ D’ is derivable using IP or CP.
At step 10, the antecedent is another logical truth.

**V. Exercises B**. Show that the following propositions are logical truths using the 18 rules, and either CP or IP.

1. $(A \supset B) \vee (B \supset A)$
2. $(P \supset Q) \supset [(P \cdot R) \supset (Q \cdot R)]$
3. $(P \cdot Q) \supset [(P \vee R) \cdot (Q \vee R)]$
4. $(A \supset B) \vee (\sim A \supset C)$
5. $(P \supset Q) \supset \{(P \supset R) \supset [P \supset (Q \cdot R)]\}$

Solutions may vary.