Introduction to Mathematical Logic

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(c) We wish a light to be controlled by two different wall switches in a room in such a way that flicking either one of these switches will turn the light on if it is off and turn it off if it is on. Construct a simple circuit to do the required job.

1.33 Determine whether the following arguments are logically correct by representing each sentence as a statement form and checking whether the conclusion is logically implied by the conjunction of the assumptions. (To do this, assign T to each assumption and F to the conclusion, and determine whether a contradiction results.)

(a) If Jones is a communist, Jones is an atheist. Jones is an atheist. Therefore, Jones is a communist.
(b) If the temperature and air pressure remained constant, there was no rain. The temperature did remain constant. Therefore, if there was rain, then the air pressure did not remain constant.
(c) If Gorton wins the election, then taxes will increase if the deficit will remain high. If Gorton wins the election, the deficit will remain high. Therefore, if Gorton wins the election, taxes will increase.
(d) If the number x ends in 0, it is divisible by 5. x does not end in 0. Hence, x is not divisible by 5.
(e) If the number x ends in 0, it is divisible by 5. x is not divisible by 5. Hence, x does not end in 0.
(f) If $a = 0$ or $b = 0$, then $ab = 0$. But $ab \neq 0$. Hence, $a \neq 0$ and $b \neq 0$.
(g) A sufficient condition for f to be integrable is that g be bounded. A necessary condition for h to be continuous is that f is integrable. Hence, if g is bounded or h is continuous, then f is integrable.
(h) Smith cannot both be a running star and smoke cigarettes. Smith is not a running star. Therefore, Smith smokes cigarettes.
(i) If Jones drove the car, Smith is innocent. If Brown fired the gun, then Smith is not innocent. Hence, if Brown fired the gun, then Jones did not drive the car.

1.34 Which of the following sets of statement forms are satisfiable, in the sense that there is an assignment of truth values to the statement letters that makes all the forms in the set true?

(a) $A \Rightarrow B$
$B \Rightarrow C$
$C \vee D \Leftrightarrow \neg B$
(b) $\neg(\neg B \vee A)$
$A \vee \neg C)$
$B \Rightarrow \neg C$
(c) $D \Rightarrow B$
$A \vee \neg B$
$\neg(D \wedge A)$
D

1.35 Check each of the following sets of statements for consistency by representing the sentences as statement forms and then testing their conjunction to see whether it is contradictory.

(a) Either the witness was intimidated or, if Doherty committed suicide, a note was found. If the witness was intimidated, then Doherty did not commit suicide. If a note was found, then Doherty committed suicide.
(b) The contract is satisfied if and only if the building is completed by 30 November. The building is completed by 30 November if and only if the electrical subcontractor completes his work by 10 November. The bank loses money if and only if the contract is not satisfied. Yet the electrical subcontractor completes his work by 10 November if and only if the bank loses money.

1.3 ADEQUATE SETS OF CONNECTIVES

Every statement form containing n statement letters generates a corresponding truth function of n arguments. The arguments and values of the function are T or F. Logically equivalent forms generate the same truth function. A natural question is whether all truth functions are so generated.

PROPOSITION 1.5

Every truth function is generated by a statement form involving the connectives $\neg$, $\wedge$ and $\vee$.

Proof

(Refer to Examples 1 and 2 below for clarification.) Let $f(x_1, \ldots, x_n)$ be a truth function. Clearly f can be represented by a truth table of 2^n rows, where each row represents some assignment of truth values to the variables $x_1, \ldots, x_n$, followed by the corresponding value of $f(x_1, \ldots, x_n)$. If $1 \leqslant i \leqslant 2^n$, let C_i be the conjunction $U_1^i \wedge U_2^i \wedge \ldots \wedge U_n^i$, where U_j^i is A_j if, in the ith row of the truth table, x_j takes the value T, and U_j^i is $\neg A_j$ if x_j takes the value F in that row. Let D be the disjunction of all those C_is such that f has the value T for the ith row of the truth table. (If there are no such rows, then f always takes the value F, and we let D be $A_1 \wedge \neg A_1$, which satisfies the theorem.) Notice that D involves only $\neg$, $\wedge$ and $\vee$. To see that D has f as its corresponding truth function, let there be given an assignment of truth values to the statement letters $A_1, \ldots, A_n$, and assume that the corrreponding assignment to the variables $x_1, \ldots, x_n$ is row k of the truth table for f. Then C_k has the value T for this assignment, whereas every other C_i has the value F. If f has the value T for row k, then C_k is a disjunct of D. Hence,

D would also have the value T for this assignment. If f has the value F for row k, then C_k is not a disjunct of D and all the disjuncts take the value F for this assignment. Therefore, D would also have the value F. Thus, D generates the truth function f.

Examples

1.

x_1	x_2	$f(x_1,x_2)$
T	T	F
F	T	T
T	F	T
F	F	T

D is $(\neg A_1 \wedge A_2) \vee (A_1 \wedge \neg A_2) \vee (\neg A_1 \wedge \neg A_2)$.

2.

x_1	x_2	x_3	$g(x_1,x_2,x_3)$
T	T	T	T
F	T	T	F
T	F	T	T
F	F	T	T
T	T	F	F
F	T	F	F
T	F	F	F
F	F	F	T

D is $(A_1 \wedge A_2 \wedge A_3) \vee (A_1 \wedge \neg A_2 \wedge A_3) \vee (\neg A_1 \wedge \neg A_2 \wedge A_3)$
$\vee (\neg A_1 \wedge \neg A_2 \wedge \neg A_3)$.

Exercise

1.36 Find statement forms in the connectives $\neg$, $\wedge$ and $\vee$ that have the following truth functions.

x_1	x_2	x_3	$f(x_1,x_2,x_3)$	$g(x_1,x_2,x_3)$	$h(x_1,x_2,x_3)$
T	T	T	T	T	F
F	T	T	T	T	T
T	F	T	T	T	F
F	F	T	F	F	F
T	T	F	F	T	T
F	T	F	F	F	T
T	F	F	F	T	F
F	F	F	T	F	T

COROLLARY 1.6

Every truth function can be generated by a statement form containing as connectives only $\wedge$ and $\neg$, or only $\vee$ and $\neg$, or only $\Rightarrow$ and $\neg$.

Proof

Notice that $\mathscr{B} \vee \mathscr{C}$ is logically equivalent to $\neg(\neg\mathscr{B} \wedge \neg\mathscr{C})$. Hence, by the second part of Proposition 1.4, any statement form in $\wedge$, $\vee$ and $\neg$ is logically equivalent to a statement form in only $\wedge$ and $\neg$ [obtained by replacing all expressions $\mathscr{B} \vee \mathscr{C}$ by $\neg(\neg\mathscr{B} \wedge \neg\mathscr{C})$]. The other parts of the corollary are similar consequences of the following tautologies:

$$\mathscr{B} \wedge \mathscr{C} \Leftrightarrow \neg(\neg\mathscr{B} \vee \neg\mathscr{C})$$
$$\mathscr{B} \vee \mathscr{C} \Leftrightarrow (\neg\mathscr{B} \Rightarrow \mathscr{C})$$
$$\mathscr{B} \wedge \mathscr{C} \Leftrightarrow \neg(\mathscr{B} \Rightarrow \neg\mathscr{C})$$

We have just seen that there are certain pairs of connectives – for example, $\wedge$ and $\neg$ – in terms of which all truth functions are definable. It turns out that there is a single connective, $\downarrow$ (joint denial), that will do the same job. Its truth table is:

A	B	$A \downarrow B$
T	T	F
F	T	F
T	F	F
F	F	T

$A \downarrow B$ is true when and only when neither A nor B is true. Clearly, $\neg A \Leftrightarrow (A \downarrow A)$ and $(A \wedge B) \Leftrightarrow ((A \downarrow A) \downarrow (B \downarrow B))$ are tautologies. Hence, the adequacy of $\downarrow$ for the construction of all truth functions follows from Corollary 1.6.

Another connective, $|$ (alternative denial), is also adequate for this purpose. Its truth table is

A	B	$A \mid B$
T	T	F
F	T	T
T	F	T
F	F	T

$A \mid B$ is true when and only when not both A and B are true. The adequacy of $|$ follows from the tautologies $\neg A \Leftrightarrow (A \mid A)$ and $(A \vee B) \Leftrightarrow ((A \mid A) \mid (B \mid B))$.

PROPOSITION 1.7

The only binary connectives that alone are adequate for the construction of all truth functions are $\downarrow$ and $|$.

Proof

Assume that $h(A,B)$ is an adequate connective. Now, if h(T,T) were T, then any statement form built up using h alone would take the value T when all its statement letters take the value T. Hence, $\neg A$ would not be definable in terms of h. So, $h(\mathrm{T,T}) = \mathrm{F}$. Likewise, $h(\mathrm{F,F}) = \mathrm{T}$. Thus, we have the partial truth table

A	B	$h(A,B)$
T	T	F
F	T	
T	F	
F	F	T

If the second and third entries in the last column are F, F or T, T, then h is $\downarrow$ or $|$. If they are F, T, then $h(A,B) \Leftrightarrow \neg B$ is a tautology; and if they are T, F, then $h(A,B) \Leftrightarrow \neg A$ is a tautology. In both cases, h would be definable in terms of $\neg$. But $\neg$ is not adequate by itself because the only truth functions of one variable definable from it are the identity function and negation itself, whereas the truth function that is always T would not be definable.

Exercises

1.37 Prove that each of the pairs $\Rightarrow$, $\vee$ and $\neg$, $\Leftrightarrow$ is not alone adequate to express all truth functions.

1.38

(a) Prove that $A \vee B$ can be expressed in terms of $\Rightarrow$ alone.
(b) Prove that $A \wedge B$ cannot be expressed in terms of $\Rightarrow$ alone.
(c) Prove that $A \Leftrightarrow B$ cannot be expressed in terms of $\Rightarrow$ alone.

1.39 Show that any two of the connectives $\{\wedge, \Rightarrow, \Leftrightarrow\}$ serve to define the remaining one.

1.40 With one variable A, there are four truth functions:

A	$\neg A$	$A \vee \neg A$	$A \wedge \neg A$
T	F	T	F
F	T	T	F

(a) With two variable A and B, how many truth functions are there ?
(b) How many truth functions of n variables are there ?

1.41 Show that the truth function h determined by $(A \vee B) \Rightarrow \neg C$ generates all truth functions.

1.42 By a *literal* we mean a statement letter or a negation of a statement letter. A statement form is said to be in *disjunctive normal form* (dnf) if it is a disjunction consisting of one or more disjuncts, each of which is a conjunction of one or more literals – for example, $(A \wedge B) \vee (\neg A \wedge C)$, $(A \wedge B \wedge \neg A) \vee (C \wedge \neg B) \vee (A \wedge \neg C)$, $A, A \wedge B$, and $A \vee (B \vee C)$. A form is in *conjunctive normal form* (cnf) if it is a conjunction of one or more conjuncts, each of which is a disjunction of one or more literals – for example, $(B \vee C) \wedge (A \vee B)$, $(B \vee \neg C) \wedge (A \vee D)$, $A \wedge (B \vee A) \wedge (\neg B \vee A)$, $A \vee \neg B$, $A \wedge B$,A. Note that our terminology considers a literal to be a (degenerate) conjunction and a (degenerate) disjunction.

(a) The proof of Proposition 1.5 shows that every statement form $\mathscr{B}$ is logically equivalent to one in disjunctive normal form. By applying this result to $\neg\mathscr{B}$, prove that $\mathscr{B}$ is also logically equivalent to a form in conjunctive normal form.
(b) Find logically equivalent dnfs and cnfs for $\neg(A \Rightarrow B) \vee (\neg A \wedge C)$ and $A \Leftrightarrow ((B \wedge \neg A) \vee C)$. [*Hint*: Instead of relying on Proposition 1.5, it is usually easier to use Exercise 1.27(b) and (c).]
(c) A dnf (cnf) is called *full* if no disjunct (conjunct) contains two occurrences of literals with the same letter and if a letter that occurs in one disjunct (conjunct) also occurs in all the others. For example, $(A \wedge \neg A \wedge B) \vee (A \wedge B)$, $(B \wedge B \wedge C) \vee (B \wedge C)$ and $(B \wedge C) \vee B$ are not full, whereas $(A \wedge B \wedge \neg C) \vee (A \wedge B \wedge C) \vee (A \wedge \neg B \wedge \neg C)$ and $(A \wedge \neg B) \vee (B \wedge A)$ are full dnfs.
   (i) Find full dnfs and cnfs logically equivalent to $(A \wedge B) \vee \neg A$ and $\neg(A \Rightarrow B) \vee (\neg A \wedge C)$.
   (ii) Prove that every non-contradictory (non-tautologous) statement form $\mathscr{B}$ is logically equivalent to a full dnf (cnf) $\mathscr{C}$, and, if $\mathscr{C}$ contains exactly n letters, then $\mathscr{B}$ is a tautology (is contradictory) if and only if $\mathscr{C}$ has 2^n disjuncts (conjuncts).
(d) For each of the following, find a logically equivalent dnf (cnf), and then find a logically equivalent full dnf (cnf),
   (i) $(A \vee B) \wedge (\neg B \vee C)$
   (ii) $\neg A \vee (B \Rightarrow \neg C)$
   (iii) $(A \wedge \neg B) \vee (A \wedge C)$
   (iv) $(A \vee B) \Leftrightarrow \neg C$
(e) Construct statement forms in $\neg$ and $\wedge$ (respectively, in $\neg$ and $\vee$ or in $\neg$ and $\Rightarrow$) logically equivalent to the statement forms in (d).

1.43 A statement form is said to be *satisfiable* if it is true for some assignment of truth values to its statement letters. The problem of determining the satisfiability of an arbitrary cnf plays an important role in the theory of computational complexity; it is an example of a so-called *$\mathscr{NP}$-complete* problem (see Garey and Johnson, 1978).

(a) Show that $\mathscr{B}$ is satisfiable if and only if $\neg\mathscr{B}$ is not a tautology.
(b) Determine whether the following are satisfiable:
   (i) $(A \vee B) \wedge (\neg A \vee B \vee C) \wedge (\neg A \vee \neg B \vee \neg C)$
   (ii) $((A \Rightarrow B) \vee C) \Leftrightarrow (\neg B \wedge (A \vee C))$
(c) Given a disjunction $\mathscr{D}$ of four or more literals: $L_1 \vee L_2 \vee \ldots \vee L_n$, let $C_1, \ldots, C_{n-2}$ be statement letters that do not occur in $\mathscr{D}$, and construct the cnf $\mathscr{E}$:

$$(L_1 \vee L_2 \vee C_1) \wedge (\neg C_1 \vee L_3 \vee C_2) \wedge (\neg C_2 \vee L_4 \vee C_3) \wedge \ldots$$
$$\wedge (\neg C_{n-3} \vee L_{n-1} \vee C_{n-2}) \wedge (\neg C_{n-2} \vee L_n \vee \neg C_1)$$

Show that any truth assignment satisfying $\mathscr{D}$ can be extended to a truth assignment satisfying $\mathscr{E}$ and, conversely, any truth assignment satisfying

$\mathscr{E}$ is an extension of a truth assignment satisfying $\mathscr{D}$. (This permits the reduction of the problem of satisfying cnfs to the corresponding problem for cnfs with each conjunct containing at most three literals.)

(d) For a disjunction $\mathscr{D}$ of three literals $L_1 \vee L_2 \vee L_3$, show that a form that has the properties of $\mathscr{E}$ in (c) cannot be constructed, with $\mathscr{E}$ a cnf in which each conjunct contains at most two literals (R. Cowen).

1.44 (*Resolution*) Let $\mathscr{B}$ be a cnf and let C be a statement letter. If C is a disjunct of a disjunction $\mathscr{D}_1$ in $\mathscr{B}$ and $\neg C$ is a disjunct of another disjunction $\mathscr{D}_2$ in $\mathscr{B}$, then a non-empty disjunction obtained by eliminating C from $\mathscr{D}_1$ and $\neg C$ from $\mathscr{D}_2$ and forming the disjunction of the remaining literals (dropping repetitions) is said to be obtained from $\mathscr{B}$ by *resolution on* C. For example, if $\mathscr{B}$ is

$$(A \vee \neg C \vee \neg B) \wedge (\neg A \vee D \vee \neg B) \wedge (C \vee D \vee A),$$

the first and third conjuncts yield $A \vee \neg B \vee D$ by resolution on C. In addition, the first and second conjuncts yield $\neg C \vee \neg B \vee D$ by resolution on A, and the second and third conjuncts yield $D \vee \neg B \vee C$ by resolution on A. If we conjoin to $\mathscr{B}$ any new disjunctions obtained by resolution on all variables, and if we apply the same procedure to the new cnf and keep on iterating this operation, the process must eventually stop, and the final result is denoted $\mathscr{R}es(\mathscr{B})$. In the example, $\mathscr{R}es(\mathscr{B})$ is:

$$(A \vee \neg C \vee \neg B) \wedge (\neg A \vee D \vee \neg B) \wedge (C \vee D \vee A) \wedge (\neg C \vee \neg B \vee D)$$
$$\wedge (D \vee \neg B \vee C) \wedge (A \vee \neg B \vee D) \wedge (D \vee \neg B)$$

(Notice that we have not been careful about specifying the order in which conjuncts or disjuncts are written, since any two arrangements will be logically equivalent.)

(a) Find $\mathscr{R}es(\mathscr{B})$ when $\mathscr{B}$ is each of the following:
 (i) $(A \vee \neg B) \wedge B$
 (ii) $(A \vee B \vee C) \wedge (A \vee \neg B \vee C)$
 (iii) $(A \vee C) \wedge (\neg A \vee B) \wedge (A \vee \neg C) \wedge (\neg A \vee \neg B)$

(b) Show that $\mathscr{B}$ logically implies $\mathscr{R}es(\mathscr{B})$.

(c) If $\mathscr{B}$ is a cnf, let $\mathscr{B}_C$ be the cnf obtained from $\mathscr{B}$ by deleting those conjuncts that contain C or $\neg C$. Let $r_C(\mathscr{B})$ be the cnf that is the conjunction of $\mathscr{B}_C$ and all those disjunctions obtained from $\mathscr{B}$ by resolution on C. For example, if $\mathscr{B}$ is the cnf in the example above, then $r_C(\mathscr{B})$ is $(\neg A \vee D \vee \neg B) \wedge (A \vee \neg B \vee D)$. Prove that, if $r_C(\mathscr{B})$ is satisfiable, then so is $\mathscr{B}$. (R. Cowen)

(d) A cnf $\mathscr{B}$ is said to be a *blatant contradiction* if it contains some letter C and its negation $\neg C$ an conjuncts. An example of a blatant contradiction is $(A \vee B) \wedge B \wedge (C \vee D) \wedge \neg B$. Prove that if $\mathscr{B}$ is unsatisfiable, then $\mathscr{R}es(\mathscr{B})$ is a blatant contradiction. [*Hint*: Use induction on the number n of letters that occur in $\mathscr{B}$. In the induction step, use (c).]

(e) Prove that $\mathscr{B}$ is unsatisfiable if and only if $\mathscr{R}es(\mathscr{B})$ is a blatant contradiction.

1.45 Let $\mathscr{B}$ and $\mathscr{D}$ be statement forms such that $\mathscr{B} \Rightarrow \mathscr{D}$ is a tautology.

(a) If $\mathscr{B}$ and $\mathscr{D}$ have no statement letters in common, show that either $\mathscr{B}$ is contradictory or $\mathscr{D}$ is a tautology.

(b) (*Craig's interpolation theorem*) If $\mathscr{B}$ and $\mathscr{D}$ have the statement letters $B_1, \ldots, B_n$ in common, prove that there is a statement form $\mathscr{C}$ having $B_1, \ldots, B_n$ as its only statement letters such that $\mathscr{B} \Rightarrow \mathscr{C}$ and $\mathscr{C} \Rightarrow \mathscr{D}$ are tautologies.

(c) Solve the special case of (b) in which $\mathscr{B}$ is $(B_1 \Rightarrow A) \wedge (A \Rightarrow B_2)$ and $\mathscr{D}$ is $(B_1 \wedge C) \Rightarrow (B_2 \wedge C)$.

1.46

(a) A certain country is inhabited only by *truth-tellers* (people who always tell the truth) and *liars* (people who always lie). Moreover, the inhabitants will respond only to *yes or no* questions. A tourist comes to a fork in a road where one branch leads to the capital and the other does not. There is no sign indicating which branch to take, but there is a native standing at the fork. What yes or no question should the tourist ask in order to determine which branch to take ? [*Hint*: Let A stand for 'You are a truth-teller' and let B stand for 'The left-hand branch leads to the capital'. Construct, by means of a suitable truth table, a statement form involving A and B such that the native's answer to the question as to whether this statement form is true will be *yes* when and only when B is true.]

(b) In a certain country, there are three kinds of people: *workers* (who always tell the truth), *businessmen* (who always lie), and *students* (who sometimes tell the truth and sometimes lie). At a fork in the road, one branch leads to the capital. A worker, a businessman and a student are standing at the side of the road but are not identifiable in any obvious way. By asking two yes or no questions, find out which fork leads to the capital (Each question may be addressed to any of the three.)

More puzzles of this kind may be found in Smullyan (1978, chap. 3; 1985, chaps 2, 4–8).

1.4 AN AXIOM SYSTEM FOR THE PROPOSITIONAL CALCULUS

Truth tables enable us to answer many of the significant questions concerning the truth-functional connectives, such as whether a given statement form is a tautology, is contradictory, or neither, and whether it logically implies or is logically equivalent to some other given statement form. The more complex parts of logic we shall treat later cannot be handled by truth tables or by any other similar effective procedure. Consequently, another