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Identity theory, §8.7

I. The identity predicate is a special predicate, with a special logic.

Consider the following logical derivation:

Superman can fly.	Fs
Superman is Clark Kent.	???
So, Clark Kent can fly.	Fc

We know the two people are the same, so anything true of one, is true of the other.
This is called the Law of the Indiscernibility of identicals: (x)(y)[(x=y) ⊃ (𝒮x≡𝒮y)]
'=' is just a special predicate, which we could write as 'Exy'
Instead, we introduce a new symbol, '='

The three rules governing Identity (ID)
1)
a=a
For any constant 'a', 'a' is identical to itself.
We can add, as a line in any proof, a statement of this form.
This only applies to constants.
2)
a=b :: b=a
Identity is commutative
This can be applied to variables, as well as to constants.
3)
𝒮a
a=b /𝒮b
If a=b, then, you may rewrite any formula containing 'a' with 'b' in the place of 'a' throughout.

II. Translation with identity
Simple ID

Clark Kent is Superman	c=s
Mary Ann Evans is George Eliot	m=g

Only and except:

John loves Mary	Ljm
John only loves Mary:	Ljm · (x)(Ljx ⊃ x=m)
Only john loves Mary:	Ljm · (x)(Lxm ⊃ x=j)
Everyone loves Mary	(x)(Px ⊃ Lxm)
Everyone except John loves Mary:	~Ljm · (x)[(Px · x≠j) ⊃ Lxm]

Note: 'x≠j' is just shorthand for '~x=j'. Negation applies to the identity predicate, and not to the objects.

Superlatives:

Degas is a better impressionist than Monet:	Id · Im · Bdm
Degas is the best impressionist:	Id · (x)[(Ix · x≠d) ⊃ Bdx]
Bill Gates is the geek with the most money	Gg · (x)[(Gx · x≠g) ⊃ Mgx]
Philadelphia is the nearest major city	Mp · (x)[(Mx · x≠p) ⊃ Npx]

Numerical operators:

There is only one applicant for the job:	$(\exists x)[Ax \cdot (y)(Ay \supset x=y)]$
There is at least one applicant for the job:	$(\exists x)Ax$
There are at least two applicants:	$(\exists x)(\exists y)[(Ax \cdot Ay) \cdot x \neq y]$
There are exactly two applicants:	$(\exists x)(\exists y)\{[(Ax \cdot Ay) \cdot x \neq y] \cdot (z)[Az \supset (z=x \vee z=y)]\}$
There is at most one applicant:	$(x)(y)[(Ax \cdot Ay) \supset x=y]$
There are at most two applicants:	$(x)(y)(z)[(Ax \cdot Ay \cdot Az) \supset (x=y \vee x=z \vee y=z)]$

Note: this statement makes no existential commitments!

III. Definite descriptions.

Consider the sentence ‘The King of France is bald’
We might translate it as ‘Bk’.
‘Bk’ is false, since there is no King of France.
So, ‘~Bk’ should be true, since it’s the negation of a false statement.
That means that ‘It’s false that the King of France is bald’ is true.
Which seems to imply that the King of France has hair.
In fact, we want both ‘The King of France is bald’ and ‘The King of France is not bald’ to be false.
So, we had better translate the sentence differently.

‘The King of France’ is a definite description
It refers to one specific object without using a name.
There are two ways to refer: by name or by description (E.g. the person who, the thing that)

Bertrand Russell’s analysis for definite descriptions:
They’re both false, due to a false presupposition.
They’re really complex statements, involving a definite description.
‘The King of France is bald’ entails three simpler expressions:
A. There is a King of France.
B. There is only one King of France.
C. That thing is bald.
So, the proposition is false because clause A is false.
‘The King of France is not bald’ is also false, for the same reason
The negation only affects the third clause.
The first is still the same, and still false.

Another example: The country called a sub-continent is India.
1. There is a country called a sub-continent
2. There is only one such country
3. That country is identical with India
So, we translate as: $(\exists x)\{(Cx \cdot Sx) \cdot (y)[(Cy \cdot Sy) \supset y=x] \cdot x=I\}$

Another example: ‘The author of Waverly was a genius’ becomes ‘$(\exists x)\{Wx \cdot (y)[Wy \supset y=x] \cdot Gx\}$’

IV. More examples.

1. Everything is identical with itself.
$(x)x=x$
2. Nothing is distinct from itself.
$(x)\sim\sim x=x$
3. The discoverer of Polonium is Polish. (Dx, Px)
$(\exists x)\{Dx \cdot (y)(Dy \supset y=x) \cdot Px]$
4. At most two persons invented the airplane. (Px, Ix)
$(x)(y)(z)[(Px \cdot Ix \cdot Py \cdot Iy \cdot Pz \cdot Iz) \supset (x=y \vee x=z \vee y=z)]$
5. There is exactly one dollar bill in my wallet. (Dx, Wx)
$(\exists x)\{(Dx \cdot Wx) \cdot (y)[(Dy \cdot Wy) \supset y=x]\}$
6. Adriana is a better dancer than Rene. (a, r, Dx, Bxy: x is better than y)

Da · Dr · Bar
7. Adriana is the best dancer.
Da · (x)[(Dx · ~x=a) ⊃ Bax]
8. There are exactly two dancers better than Adriana.
(∃x)(∃y){(Dx · Bxa · Dy · Bya · ~x=y) · (z)[(Dz · Bza) ⊃ (z=x ∨ z=y)]}

V. Exercises A. Translate the following:

1. Everything is identical with something.
2. There are prime numbers. (Px, Nx)
3. Two is the only even prime number. (t, Ex, Px, Nx)
4. There is exactly one even prime number.
5. There are at least two odd prime numbers. (Ox, Px, Nx)
6. All prime numbers are odd except the number two.
7. The murderer was Colonel Mustard. (m, Mx)
8. There is at least one dancer better than Rene.
9. Rene is the worst dancer.
10. Goliath is the tallest human. (g, Hx, Txy: x is taller than y)

VI. Derivations
Consider the original problem:
Superman can fly.
Superman is Clark Kent
∴ Clark Kent can fly.

1. Fs
2. s=c / Fc
3. Fc 1, 2, ID
QED

Using the commutative Id rule:
1. a=b ⊃ j=k
2. b=a
3. Fj / Fk
4. a=b 2, Id
5. j=k 1, 4, MP
6. Fk 3, 5, Id
QED

To derive the negation of an identity statement, you must use (IP):
1. Rm
2. ~Rj / m≠j
*3. m=j
*4. Rj
*5. Rj · ~Rj
6. m≠j
QED

Using the reflexive Id rule:
1. (x)(~Gx ⊃ x≠d) / Gd
*2. ~Gd AIP
*3. ~Gd ⊃ d≠d 1, UI
*4. d=d Id
*5. d≠d 3, 2, MP
*6. d=d · d≠d 4, 5, Conj

7. Gd
QED

Proof with existential conclusion:
1. Rab
2. (∃x)~Rxb /(∃x)~x=a
3. ~Rcb 2, EI
*4. c=a AIP
*5. Rcb 1, Id
*6. Rcb · ~Rcb 5, 3, Conj
7. ~c=a 4-6, IP
8. (∃x)~x=a 7, EG
QED

A final example: The Faulkner scholar at Swarthmore is very learned. Therefore, all Faulkner scholars at Swarthmore are very learned.

1. (∃x){{(Sx · Fx) · (y)[(Sy · Fy) ⊃ x=y]} · Lx} / (x)[(Sx · Fx) ⊃ Lx]
*2. ~(x)[(Sx · Fx) ⊃ Lx] AIP
*3. (∃x)~[(Sx · Fx) ⊃ Lx] 2, CQ
*4. ~[(Sa · Fa) ⊃ La] 3, EI
*5. ~[~(Sa · Fa) ∨ La] 4, Impl
*6. (Sa · Fa) · ~La 5, DM, DN
*7. {(Sb · Fb) · (y)[(Sy · Fy) ⊃ b=y]} · Lb 1, EI (to b)
*8. (y)[(Sy · Fy) ⊃ b=y] 7, Simp, Com, Simp
*9. (Sa · Fa) ⊃ b=a 8, UI (to a)
*10. Sa · Fa 6, Simp
*11. b=a 9, 10, MP
*12. Lb 7, Simp
*13. La 12, 11, Id
*14. ~La 6, Com, Simp
*15. La · ~La 13, 14, Conj
16. (x)[(Sx · Fx) ⊃ Lx] 2-15, IP
QED

VI. Exercises B. Derive the conclusions of each of the following arguments.
1)
1. (x)(Dx ⊃ Ex)
2. Da
3. a=b / Eb

2)
1. (x)(Ax ⊃ Bx)
2. ~Bf
3. Ae / f≠e

3)
1. (x)(Hx ⊃ Jx)
2. (x)(Kx ⊃ Lx)
3. Hd · Kc
4. c=d / Jc · Ld

4)
1. (x)(y)(x=y)
2. (x)Mxx / Mab

5)
1. (x)[(∃y)Kxy ⊃ (∃z)Kzx]
2. (∃x)(Kxg · x=b) / (∃z)Kzb

6)
1. (∃x)Hx
2. (x)(y)[(Hx · Hy) ⊃ x=y] / (∃x)[Hx · (y)(Hy ⊃ x=y)]

VII. Solutions.

Answers to Exercises A.
1. (x)(∃y)x=y
2. (∃x)(Px · Nx)
3. Et · Pt · Nt · (x)[(Ex · Px · Nx) ⊃ x=t]
4. (∃x){(Ex · Px · Nx) · (y)[(Ey · Py · Ny) ⊃ y=x]}
5. (∃x)(∃y)(Ox · Px · Nx · Oy · Py · Ny · ~x=y)
6. (x)[(Px · Nx · ~x=t) ⊃ Ox]
7. (∃x)[Mx · (y)(My ⊃ y=x) · x=m]
8. Dr · (∃x)(Dx · Bxr)
9. Dr · (x)[(Dx · ~x=r) ⊃ Bxr]
10. Hg · (x)[(Hx · ~x=g) ⊃ Tgx]
Or, using definite descriptions: (∃x){(y)[(Hy · ~y=x) ⊃ Txy] · (z){(y)[(Hy · ~y=z) ⊃ Tzy] ⊃ z=x} · x=g}

A solution to Exercise B.6:

1. (∃x)Hx	
2. (x)(y)[(Hx · Hy) ⊃ x=y]	/(∃x)[Hx · (y)(Hy ⊃ x=y)]
3. Ha	1, EI
*4. ~(∃x)[Hx · (y)(Hy ⊃ x=y)]	AIP
*5. (x)~[Hx · (y)(Hy ⊃ x=y)]	4, CQ
*6. (x)[~Hx ∨ ~(y)(Hy ⊃ x=y)]	5, DM
*7. ~Ha ∨ ~(y)(Hy ⊃ a=y)	6, UI
*8. ~(y)(Hy ⊃ a=y)	7, 3, DN, DS
*9. (∃y) ~(Hy ⊃ a=y)	8, CQ
*10. ~(Hb ⊃ a=b)	9, EI
*11. ~(~Hb ∨ a=b)	10, Impl
*12. Hb · ~a=b	11, DM, DN
*13. Hb	12, Simp
*14.~a=b	12, Com Simp
*15. (y)[(Ha · Hy) ⊃ a=y]	2, UI
*16. (Ha · Hb) ⊃ a=b	15, UI
*17. Ha · Hb	3, 13, Conj
*18. a=b	16, 17, MP
*19. a=b · ~a=b	18, 14, Conj
20. (∃x)[Hx · (y)(Hy ⊃ x=y)]	4-19, IP