Class #4 - Frege's Projects

I. Frege and the Philosophy of Mathematics

Gottlob Frege's influence extends into almost every area of contemporary philosophy: logic, mathematics, language, science, mind, metaphysics, epistemology.
Despite the breadth of his work, all of it derives from an attempt to provide a foundation for mathematics.
The language revolution is merely a side project for Frege, an attempt to work out the bugs in the formal language he devised to proved a syntactic, algorithmic definition of logical consequence.
We will not spend much time on either Frege's formal language, which is a mere notational variant of contemporary systems of symbolic logic, or his philosophy of mathematics, which was easily as profoundly influential as his philosophy of language.
But, it will be important to understand a few central aspects of his work in these two areas.

The problems to which Frege was responding, the reasons for developing his formal language, were some odd results in nineteenth-century mathematics.
For nearly two hundred years, mathematicians had worked with the calculus of Newton and Leibniz.
The calculus allowed mathematicians to find the area under a curve by dividing the area into infinitely many infinitely small areas.
Working with infinity, both small and large, worried mathematicians, even if the resulting calculations were successful.
Infinity was seen as mainly a theological concept, the primary attribute of God, and not a properly scientific concept.
To make matters worse, Cantor, in the mid-nineteenth century discovered a proof that there are different sizes of infinity.
Indeed, it turns out that there are infinitely many difference sizes of infinity.
Cantor's claim struck many mathematicians as absurd, but they could not find a flaw in his logic.

Similarly worrisome, for Frege, were recent developments in non-Euclidean geometry.
According to Euclid's controversial parallel postulate, given a line, and a point outside that line, there is one and only line which passes through the point parallel to the given line.
(Actually, that formulation of Euclid's fifth postulate is called Playfair's postulate; Euclid worked with an equivalent but different formulation.)
For centuries, Euclid's postulate seemed too controversial to be adopted as an unproven axiom.
Yet the parallel postulate resisted derivation from the other axioms.
Research into the parallel postulate had led, by the early nineteenth century, not to its proof, but to the realization that one could develop various different consistent, yet non-Euclidean, systems of geometry by denying the postulate.
If one adopts the claim that there are no lines through the given point parallel to the given line, the geometry of spheres results.
The claim that there are more than one parallel line leads to systems of hyperbolic geometry.
In the twentieth century, hyperbolic geometry was shown not only to be consistent, but to be the correct geometry for space-time, according to the theory of relativity.
Again, Frege and the mathematicians considering the counter-intuitive results of non-Euclidean geometries worried that the laws of logic were being flouted.
Philosophers began to think more carefully about both the foundations of mathematics and the notion of logical consequence: what follows from what.

Frege began a project to show that all of mathematics can be reduced to logic.
Logic is obvious and uncontroversial.
If Frege could show that mathematics were just logic in disguise, then it too would be shown to be uncontroversial.
This project came to be known as logicism.
To show that mathematics is just logic, Frege first had to regiment the notion of a proof.
In order to do so, he produced a formal language for such proofs.
Frege called this language his *Begriffsschrift*, or concept-writing.

Using the formal language in the *Begriffsschrift*, Frege was able to refine the notion of logical truth, truths which depend merely on language.
If truth has something to do with language, and something to do with the world, it seems to follow, as Quine puts it, that some truths will depend on language alone: all bachelors are unmarried, if p then p, perhaps 2+2=4.
Hume had called such truths relations of ideas, and declared that they follow from the principle of non-contradiction.
Kant, similarly, declared such truths *a priori*.
Frege provided a formal method for characterizing these truths.
Whether there are such truths, and whether there are meanings at all, is the subject of Quine's "Two Dogmas of Empiricism," on which we will spend time later in the term.

Frege's formal language turned out to have a much broader application than just to mathematics.
Indeed, my claim that all of Frege's work centers on his mathematical project is contentious.
Michael Dummett, for example, claims that Frege's focus on the philosophy of language is consequent to his interest in thought.

> For Frege, as for all subsequent analytical philosophers, the philosophy of language is the foundation of all other philosophy because it is only by the analysis of language that we can analyse thought. Thoughts differ from all else that is said to be among the contents of the mind in being wholly communicable: it is of the essence of thought that I can convey to you the very thought I have, as opposed to being able to tell you merely something about what my thought is like. It is of the essence of thought, not merely to be communicable, but to be communicable, without residue, by means of language. In order to understand thought, it is necessary, therefore, to comprehend the means by which thought is expressed (Dummett, *Truth and Other Enigmas* 442).

Whether Frege pursued the philosophy of language in order primarily to understand the essence of thought or to facilitate his logicist program in the philosophy of mathematics is immaterial.
Either way, the linguistic revolution was begun.
Philosophers, encouraged by the clarity and precision of Frege's new formal language, developed hopes of using it as canonical in all proper scientific discourse.

Frege's work, centered around the logicist project, developed as follows.
In 1879, he published the *Begriffsschrift*, founding modern mathematical logic.
In 1884, he published the *Grundlagen* (Foundations of Arithmetic), in which he described and defended logicism.
In 1892, he published "Über Sinn und Bedeutung" ("On Sense and Reference"), the seminal work of the revolution, and its companion article, "Über Begriff und Gegenstand" ("On Concept and Object").

The *Grundgesetze*, in two (of three planned) volumes (1893 and 1903), did some of the technical work promised in the *Grundlagen*.
Just as the second volume appeared, Bertrand Russell discovered a paradox at the core of Frege's formal theory.
Frege's project never recovered.
He despaired of completing the work and never wrote the intended third volume.
Russell, undeterred, developed a way to work around the paradoxes; he and Alfred North Whitehead produced the massive *Principia Mathematica* in three volumes (1910, 1912, and 1913), pursuing the logicist project.
Frege's "The Thought" was originally published later in his life, in 1918.

To understand fully the context for Frege's work on language, I have asked you to read small portions of the prefaces to both the *Begriffsschrift* and the *Grundlagen*.

II. Frege and Formal Systems

The importance of the *Begriffsschrift*, for our purposes, is its expression of Frege's desire to hold a microscope up to our language.
This desire is manifest in the construction of formal theories.
Contemporary linguistic theories rely on formal theories of syntax.
These formal theories trace directly to Frege's work on logic.
Once Frege introduced formal languages, it was a short step to the distinction between syntax, or the formal properties of those languages, and semantics, or the content of the language.
Linguists work almost exclusively on syntax, looking across languages.
In symbolic logic, we use rules of inference that are syntactically specified, making sure all gaps are filled.

In studying formal systems, there are several things you can do:

1. Construct a language
2. State some axioms, or basic principles, for a theory
3. Provide rules of inference, to derive other theorems
4. Interpret, or model, the theory

To get a feel for working with formal systems, consider the following system, found in *Gödel, Escher, Bach: An Eternal Golden Braid*, by Douglas Hofstadter.
We start by specifying the language.
The particles of the MU system are 'M', 'I', and 'U'.
In the MU system, any string of 'M's, 'I's, and 'U's is a string of the MU system.
So: MIU, UMI, and MMMUMUUUMUMMU are all strings.
Analogously, we can take the words of English to be the basic particles.
Then, only some strings, the grammatical sentences, will be formulas of the language.
So far, we have only specified the language.
We can proceed to write a theory, in that language.

An axiom is a basic assumption of a theory.
A theorem is any string which is either an axiom, or follows from the axioms by using some combination

of the rules of inference.
In the MIU system, we will only be interested in theorems, just as we might be interested only in the true sentences of English.

The MIU system takes only one axiom: MI.
There are four rules of inference:

R1. If a string ends in I you can add U.
R2. From Mx, you can infer Mxx.
That is, you can repeat whatever follows an M.
R3. If III appears in that order, then you can replace the three Is with a U
R4. UU can be dropped from any theorem.

Starting with MI, we can derive various theorems:

Statement	Justification
1. MI	Axiom
2. MIU	From Step 1 and R1
3. MII	1, R2
4. MIIII	3, R2
5. MIU	4, R3
6. MUI	4, R3
7. MIIIIIIII	4, R2
8. MIUUI	7, R3
9. MII	8, R4

For a small challenge, try to derive 'MIIIII' (five I's).
For a more serious challenge, try to derive 'MU'.
For help on deriving 'MU', see Hofstadter's book, pp 259-261.
Do not spend much time on this puzzle without consulting Hofstadter, who provides hints and a solution!

Notice that in the MU system, there is no indication what any of the theorems or strings mean.
The whole system is based on abstract, symbolic manipulation, according to specific, algorithms.
Natural language has syntactic elements.
The grammaticality of a sentence, for example, does not depend on the content of the terms, but on the categories to which the terms belong, and the rules for constructing sentences.
We will look briefly at Chomsky's revolutionary claim about grammaticality, that there is a universal grammar built in to the structure of our brains, at the end of the term.

In contrast to linguists, philosophers of language are less interested in syntax than they are in semantics and pragmatics.
Semantics concerns the interpretations of our language.
When we interpret statements of a formal language, as when we translate between logic and natural language, we are doing semantics.
Jabberwocky, for example, could be taken as devoid of content.
But, we impose some interpretations on the terms.
Martinich has an introduction to syntax and semantics in §VII - §VIII of his introduction to *The Philosophy of Language*.

If you haven't taken symbolic logic, you should spend some time with it.
If you have taken symbolic logic, you should spend a bit of time with it, too, but it will be easier to do so.

The term 'semantics' can be used to refer to two different kinds of theories: a semantic theory and a theory of truth.
A semantic theory for a particular language may take different forms.
It should yield an infinite number of theorems of the following form.

S1 'snow is white' means-in-English that snow is white
S2 'grass is green' means-in-English that grass is green

The theorems of semantic theory belong to a meta-language in which the semantic theory for an object language is written.
For S1 and S2, the meta-language and the object language are identical.
We can better see the difference between an object language and a meta-language by considering a semantic theory for a separate language.
Such a theory will include, in the meta-language, theorems like S3 and S4.

S3 'la nieve es blanca' means-in-Spanish that snow is white
S4 'la hierba es verde' means-in-Spanish that grass is green

Notice the scare-quotes on the left sides of S1 - S4.
These indicate that we are merely mentioning the sentence inside the scare quotes, rather than using it.
When the object language and the meta-language are different, it is easy to distinguish use and mention.
It is more difficult to distinguish them when the object language and meta-language are the same, so we use scare quotes, or any of a number of devices to display a term.

Truth theories ordinarily start with a minimal condition which we call the T-schema, or Convention T, following Tarski.

TS p is true iff x

In TS, 'p' is the name of any sentence, and x are the truth conditions of that sentence.
We can use TS to specify the truth conditions for any sentence.

T1 'snow is white' is true if and only if snow is white
T2 'grass is green' is true if and only if grass is green
T3 '2+2=4' is true if and only if 2+2=4
T4 'Barack Obama is president' is true if and only if the husband of Michelle Obama and father of Sasha Obama and Malia Obama is head of the executive branch of the United States of America.

Note that, as in T4, the truth conditions on the right need not be expressed in the same terms as the sentence on the left.
We can even use a different language for the sentence and for its truth conditions.

T5 'El gato está en el alfombrilla' is true iff the cat is on the mat.

Notice that you could understand the truth conditions of 7 without understanding the meaning of the Spanish sentence on the left side.
We will talk more about semantic theories and truth theories during the course.

In addition to syntax, semantics, and truth, some philosophers of language are also interested in the difference between what is said and what is communicated.
The study of what gets communicated is called pragmatics.
Pragmatics is the study of what kinds of acts can be performed with language, in addition to the meaning of one's words.

For example, consider contrastive stress in which we emphasize certain words in a sentence in order to alter the use of the sentence.

> Humpty Dumpty was sitting with his legs crossed, like a Turk, on the top of a high wall - such a narrow one that Alice quite wondered how he could keep his balance - and, as his eyes were steadily fixed in the opposite direction, and he didn't take the least notice of her, she thought he must be a stuffed figure after all.
> "And how exactly like an egg he is!" she said aloud, standing with her hands ready to catch him, for she was every moment expecting him to fall.
> "It's *very* provoking," Humpty Dumpty said after a long silence, looking away from Alice as he spoke, "to be called an egg - *very*!"
> "I said you *looked* like an egg, Sir," Alice gently explained. "And some eggs are very pretty, you know," she added, hoping to turn her remark into a sort of a compliment.

The most important paper on pragmatics is Grice's "Logic and Conversation" which I have not assigned, but which is available on the course website.
Frege makes some brief comments on pragmatics in "The Thought," p 295 et seq., mainly to show their independence from meaning.
We will not pursue that topic very far.

Linguists have made phenomenal advances in syntactic theories since Frege's time.
Work on semantic theories has been less productive.
We will spend more time looking at formal theories of truth later.
Let's get back to the second step of Frege's logicist project, the *Grundlagen*.

III. Frege's Three Principles

Frege developed the *Begriffsschrift* in order to show that all of mathematics could be reduced to logic.
He explored this logicist thesis rhetorically, philosophically, in the *Grundlagen*.
Again, the *Grundlagen* is not our focus.
But it is instructive to note three guidelines which Frege presents in the introduction to that work.

> FG1. Always to separate sharply the psychological from the logical, the subjective from the objective;
> FG2. Never to ask for the meaning of a word in isolation, but only in the context of a proposition;
> FG3. Never to lose sight of the distinction between concept and object (Frege, *Grundlagen x)*.

We can see in FG1 Frege's desire to distance his work in language from that of the Moderns and the nineteenth-century idealists.
Frege wants a logical, objective theory of mathematics and language, not a psychological one, like Locke's.
With Meinong and Mill, Frege pursues a standard, objective theory of truth.

FG2 has come to be called the context principle.
The idea behind FG2 is that the basic unit of meaning is not the single term, but the sentence.
'Theaetetus' itself has no meaning independently of how we use that term in an assertion.
There might be many Theaetetuses, for example.
We will see this principle, a bit, in Quine's work.
Even a baby's use of the single term 'Mama', which might be taken as a counter-example to FG2, is best taken as an assertion, 'There is my mother', rather than as a mere label.
Frege's argument for FG2, in the *Grundlagen*, is that if we take terms, like 'two', as labels, outside of an assertion, we end up looking for the referents of such terms.
Since there are no twos in the world, we end up thinking that 'two' refers to my idea of a two.
We end up believing that the referents of my terms are ideas, and we are back with Locke, Berkeley, Hume, and Kant stuck in our phenomenal worlds.
Giving up FG2 leads back to idealism.

FG2 should not be interpreted so as to contradict the claim that (human) languages are essentially compositional.
Compositionality requires that the meanings of the whole proposition are constructed out of the meanings of the parts, and that the truth of the proposition depends on the truth of its parts.
The context principle demands that the meanings of the parts are somehow dependent on the meanings of the whole.
Compositionality is a fundamental principle of both logic and formal theories of language.
Compositionality and the context principle are in tension, but they can be seen to work together.
There is a link to a paper by Oystein Linnebo on our website which attempts to resolve the tension.

Frege and Mill disagree concerning FG3.
Mill, indeed, thinks of 'blue' as the name of a thing, a general name, rather than the name of a concept.
Frege spends a lot of the *Grundlagen* attacking Mill's empiricist philosophy of mathematics.
FG3 is a central part of both Frege's attack on Mill and his positive account of numbers, but these are not our concern here.

With the background of Frege's logicist project, and its emphasis on formal language, as well as his three principles FG1 - FG3, let's proceed to look at Frege's philosophy of language.
Frege's philosophy of language is contained mainly in three papers: the late "The Thought", and the earlier "On Sense and Reference" and its companion piece (which I did not assign) "On Concept and Object."

IV. Frege Against Idealism

Frege asks a lot of important questions in both "The Thought" and "On Sense and Reference."
Some of his answers and some of his exposition, in retrospect, are confused.
For example, Frege confuses semantic theories and truth theories.

He was the first person to look at the form and structure of language in the kind of detail, and with the kinds of philosophical questions in mind, that he does.
So we can excuse the confusion.
Still, we will have to try to separate the semantic questions from the questions about truth, if only to keep them clear.

Much of the later part of "The Thought" is not especially relevant to our concerns.
Frege was working in response to nineteenth-century idealism.
We can see his objections to idealism clearly on pp 303-308, where he takes on the enormous question of how we know that anything exists beyond my own consciousness.

> If we call what happens in our consciousness idea, then we really experience only ideas but not their causes (304).
> It is inconceivable that I should be boxed into myself in this way to infinity (305).
> If man could not think and could not take something of which he was not the bearer as the object of his thought, he would have an inner world but no outer world. But may this not be based on a mistake? (306)
> Not everything that can be the object of my understanding is an idea (307).

Frege's central argument against idealism is that it leads to a regressive absurdity.
Every idea requires a thinker.
But the principles underlying idealism allow the inference only to further ideas.
So those principles that I can know only my own ideas must be flawed.

Frege also provides a secondary argument against idealism.
If idealism were true, then psychology would be the most fundamental science.
But, psychology is subordinate to mathematics.
So, idealism is false.

> Not everything is an idea. Otherwise psychology would contain all the sciences within it or at least it would be the highest judge over all the sciences. Otherwise psychology would rule over logic and mathematics. But nothing would be a greater misunderstanding of mathematics than its subordination to psychology. Neither logic nor mathematics has the task of investigating minds and the contents of consciousness whose bearer is a single person (Frege, "The Thought" 308).

Furthermore, if we pursue logic psychologically, rather than objectively, we will end up unable to distinguish truth from falsity.
Psychology can tell us about how our beliefs are formed, but not whether they are true.

> Error and superstition have causes just as much as genuine knowledge. The assertion both of what is false and of what is true takes place in accordance with psychological laws. A derivation from these and an explanation of a mental process that terminates in an assertion can never take the place of a proof of what is asserted (Frege, "The Thought" 290).

But we have said enough about idealism.
Our concern is with Frege's positive proposal about the objects of language.