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Zeno's Paradox #1. The Achilles

Achilles, who is the fastest runner of antiquity, is racing to catch the tortoise that is slowly crawling away from him. Both are moving along a linear path at constant speeds. In order to catch the tortoise, Achilles will have to reach the place where the tortoise presently is. However, by the time Achilles gets there, the tortoise will have crawled to a new location. Achilles will then have to reach this new location. By the time Achilles reaches that location, the tortoise will have moved on to yet another location, and so on forever. Zeno claims Achilles will never catch the tortoise. He might have defended this conclusion in various ways—by saying it is because the sequence of goals or locations has no final member, or requires too much distance to travel, or requires too much travel time, or requires too many tasks. However, if we do believe that Achilles succeeds and that motion is possible, then we are victims of illusion, as Parmenides says we are.

It won't do to react and say the solution to the paradox is that there are biological limitations on how small a step Achilles can take. Achilles' feet aren't obligated to stop and start again at each of the locations described above, so there is no limit to how close one of those locations can be to another. It is best to think of the change from one location to another as a movement rather than as incremental steps requiring halting and starting again. Zeno is assuming that space and time are infinitely divisible; they are not discrete or atomistic. If they were, the Paradox's argument would not work.

One common complaint with Zeno's reasoning is that he is setting up a straw man because it is obvious that Achilles cannot catch the tortoise if he continually takes a bad aim toward the place where the tortoise is; he should aim farther ahead. The mistake in this complaint is that even if Achilles took some sort of better aim, it is still true that he is required to go to every one of those locations that are the goals of the so-called "bad aims."

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Standard Solution to Zeno's Paradox #1. The Achilles

Achilles' path [the path of some dimensionless point of Achilles' body] is a linear continuum and so is composed of an actual infinity of points. (An actual infinity is also called a "completed infinity" or "transfinite infinity.") Achilles travels a distance d_1 in reaching the point x_1 where the tortoise starts, but by the time Achilles reaches x_1, the tortoise has moved on to a new point x_2. When Achilles reaches x_2, having gone an additional distance d_2, the tortoise has moved on to point x_3, and so forth. This sequence of non-overlapping distances (or intervals or sub-paths) is an actual infinity, but happily the sum of its terms $d_1 + d_2 + d_3 + \ldots$ is a finite distance that Achilles can readily complete while moving at a constant speed, because the sequence of sub-paths converges fast enough. In his argument, Zeno drew the incorrect conclusion that the sequence cannot be completed because it has no final member (or requires too much distance to travel, or requires too much travel time).

Zeno’s Paradox #2. The Dichotomy (Racetrack)

In his Progressive Dichotomy Paradox, Zeno argued that a runner will never reach a fixed goal along the racetrack. The reason is that the runner must first reach half the distance to the goal, but when there he must then cross half the remaining distance, then half of the new remainder, and so on. If the goal is one meter away, the runner must cover a distance of 1/2 meter, then 1/4 meter, then 1/8 meter, and so on ad infinitum. The runner cannot reach the final goal, says Zeno.

The runner will not reach the final goal for four reasons: (1) there is not enough time, (2) there is too far to run, (3) the actually infinite sequence has no final member, and (4) there are so many tasks to complete.

The problem of the runner getting to the goal can be viewed from a different perspective. According to the Regressive version of the Dichotomy Paradox, the runner cannot even take a first step. Here is why. Any step may be divided conceptually into a first half and a second half. Before taking a full step, the runner must take a 1/2 step, but before that he must take a 1/4 step, but before that a 1/8 step, and so forth ad infinitum, so Achilles will never get going. The original distance between the runner and the goal is not relevant.

The Dichotomy paradox, in either its Progressive version or its Regressive version, assumes for the sake of simplicity that the runner’s positions are point places. Actual runners take up some space. But this is not a controversial assumption because Zeno could have reconstructed his paradox by speaking of the point places occupied by the tip of the runner’s nose.

Philosophy 110W: Introduction to Philosophy | Hamilton College
Spring 2012 | Russell Marcus

Standard Solution to Zeno's Paradox #2. The Dichotomy (Racetrack)

The runner reaches the points 1/2 and 3/4 and 7/8 and so forth on the way to his goal, but under the influence of Bolzano and Cantor, who developed the first theory of sets, the set of those points is no longer considered to be potentially infinite. It is an actually infinite set of points abstracted from a continuum of points–in the contemporary sense of "continuum" at the heart of calculus. And the ancient idea that the actually infinite series of path lengths 1/2 + 1/4 + 1/8 + … is infinite had to be rejected in favor of the new theory that it converges to 1.

Zeno's Paradox #3. The Arrow

A moving arrow must occupy a space equal to itself at any moment. That is, at any moment it is at the place where it is. But places do not move. So, if at each moment, the arrow is occupying a space equal to itself, then the arrow is not moving at that moment because it has no time in which to move; it is simply there at the place. The same holds for any other moment during the so-called "flight" of the arrow. So, the arrow is never moving. Similarly, nothing else moves.

Standard Solution to Zeno's Paradox #3. The Arrow

The Standard Solution to the Arrow Paradox uses the "at-at" theory of motion, which says that being at rest involves being motionless at a particular point at a particular time, and that being in motion does, too. The difference between rest and motion has to do with what is happening at nearby moments. An object cannot be in motion in an instant, but it can be in motion at an instant in the sense of having a speed at that instant, provided the object occupies different positions at times before or after that instant so that the instant is part of a period in which the arrow is continuously in motion.

Zeno would have balked at the idea of motion at an instant, believing that all motion occurs only over a duration of time, and that durations divide into intervals but never into indivisible instants. However, in calculus, speed at an instant (instantaneous velocity) is the limit of the speed over an interval as the length of the interval tends to zero. The derivative of position x with respect to time t, namely dx/dt, is the arrow's speed, and it has non-zero values at specific places at specific instants during the flight, contra Zeno. The speed during an instant or in an instant, which is what Zeno is calling for, would be 0/0 and so is undefined. Using these modern concepts, Zeno cannot successfully argue that at each moment the arrow is at rest or that the speed of the arrow is zero at every instant. Therefore, advocates of the Standard Solution conclude that Zeno's Arrow Paradox has a false, but crucial, assumption and so is unsound.

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Zeno's Paradox #4. Limited and Unlimited

Suppose there exist many things. Then there will be a definite or fixed number of those many things, and so they will be "limited." But if there are many things, say two things, then they must be distinct, and to keep them distinct there must be a third thing separating them. So, there are three things. But between these, …. In other words, things are dense and there is no definite or fixed number of them, so they will be "unlimited." This is a contradiction, because the plurality would be both limited and unlimited.

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Standard Solution to Zeno's Paradox #4. Limited and Unlimited

The weakness of Zeno's argument can be said to lie in the assumption that "to keep them distinct, there must a third thing separating them." Zeno would have been correct to say that between any two physical objects that are separated in space, there is a place between them, because space is dense, but he is mistaken to claim that there must be a third physical object there between them. Two objects can be distinct at a time simply by one having a property the other does not have.

Zeno's Paradox #5. Large and Small

Suppose there exist many things. These things must be composed of parts which are not themselves pluralities. Yet things that are not pluralities cannot have a size or else they'd be divisible into parts and thus be pluralities themselves.

But the parts of pluralities are so large as to be infinite. The parts cannot be so small as to have no size since adding such things together would never contribute anything to the whole so far as size is concerned. So, the parts have some non-zero size. If so, then each of these parts will have two spatially distinct sub-parts, one in front of the other. Each of these sub-parts also will have a size. The front part, being a thing, will have its own two spatially distinct sub-parts, one in front of the other; and these two sub-parts will have sizes. Ditto for the back part. And so on without end. A sum of all these sub-parts would be infinite. Therefore, each part of a plurality will be so large as to be infinite.

Thus every part of any plurality is both so small as to have no size but also so large as to be infinite.

Standard Solution to Zeno's Paradox #5. Large and Small

There are many errors here in Zeno's reasoning, according to the Standard Solution. He is mistaken at the beginning when he says, "If there is a plurality, then it must be composed of parts which are not themselves pluralities." A university is an illustrative counterexample. A university is a plurality of students, but we need not rule out the possibility that a student is a plurality. What's a whole and what's a plurality depends on our purposes. When we consider a university to be a plurality of students, we consider the students to be wholes without parts. But for another purpose we might want to say that a student is a plurality of biological cells. Zeno is confused about this notion of relativity, and about part-whole reasoning.

A second error occurs in arguing that the each part of a plurality must have a non-zero size. In 1901, Henri Lebesgue showed how to properly define the measure function so that a line segment has nonzero measure even though (the singleton set of) any point has a zero measure. Lebesgue's theory is our current civilization's theory of measure, and thus of length, volume, duration, mass, voltage, brightness, and other continuous magnitudes.

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Zeno's Paradox #6. Infinite Divisibility

Imagine cutting an object into two non-overlapping parts, then similarly cutting these parts into parts, and so on until the process of repeated division is complete. Assuming the hypothetical division is "exhaustive" or does comes to an end, then at the end we reach what Zeno calls "the elements." Here there is a problem about reassembly. There are three possibilities. (1) The elements are nothing. In that case the original objects will be a composite of nothing, and so the whole object will be a mere appearance, which is absurd. (2) The elements are something, but they have zero size. So, the original object is composed of elements of zero size. Adding an infinity of zeros yields a zero sum, so the original object had no size, which is absurd. (3) The elements are something, but they do not have zero size. If so, these can be further divided, and the process of division was not complete after all, which contradicts our assumption that the process was already complete. In summary, there were three possibilities, but all three possibilities lead to absurdity. So, objects are not divisible into a plurality of parts.

Standard Solution to Zeno's Paradox #6. Infinite Divisibility

We first should ask Zeno to be clearer about what he is dividing. Is it concrete or abstract? When dividing a concrete, material stick into its components, we reach ultimate constituents of matter such as quarks and electrons that cannot be further divided. These have a size, a zero size (according to quantum electrodynamics), but it is incorrect to conclude that the whole stick has no size if its constituents have zero size. [Due to the forces involved, point particles have finite "cross sections," and configurations of those particles, such as atoms, do have finite size even if composed of zero-size quarks and electrons.] So, Zeno is wrong here.

On the other hand, is Zeno dividing an abstract path or trajectory? Let's assume he is, since this produces a more challenging paradox. If so, then choice (2) above is the one to think about. It's the one that talks about addition of zeroes. Let's assume the object is one-dimensional, like a path. According to the Standard Solution, this "object" that gets divided should be considered to be a continuum with its elements arranged into the order type of the linear continuum, and we should use Lebesgue's notion of measure to find the size of the object. The size (length, measure) of a point-element is zero, but Zeno is mistaken in saying the total size (length, measure) of all the zero-size elements is zero. The size of the object is determined instead by the difference in coordinate numbers assigned to the end points of the object. An object extending along a straight line that has one of its end points at one meter from the origin and other end point at three meters from the origin has a size of two meters and not zero meters. So, there is no reassembly problem, and a crucial step in Zeno's argument breaks down.

Zeno's Paradox #7. The Grain of Wheat

Version 1: When a bushel of wheat grains crashes to the floor, it makes a sound. Since the bushel is composed of individual grains, each individual grain also makes a sound, as should each thousandth part of the grain, and so on to its ultimate parts. But this result contradicts the fact that we actually hear no sound for portions like a thousandth part of a grain, and so we surely would hear no sound for an ultimate part of a grain. Yet, how can the bushel make a sound if none of its ultimate parts make a sound?

Version 2: When a bushel of wheat grains crashes to the floor, it makes a sound. The bushel is composed of individual grains, so they, too, make an audible sound. But if you drop an individual millet grain or a small part of one or an even smaller part, then eventually your hearing detects no sound, even though there is one. Therefore, you cannot trust your sense of hearing.

Standard Solution to Zeno's Paradox #7. The Grain of Wheat

Zeno mistakenly assumes that there is no lower bound on the size of something that can make a sound. There is no problem, we now say, with parts having very different properties from the wholes that they constitute. The iterative rule is initially plausible but ultimately not trustworthy, and Zeno is committing both the fallacy of division and the fallacy of composition.

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Zeno's Paradoxes[1]

Work groups questions:

1. What assumptions about space, motion, or time does Zeno make? Are these assumptions commonsensical? Are they defensible?

2. Can the paradox be solved by abandoning one or more assumptions?

3. Consider the standard solution. Are there alternatives?

Base group questions

1. How are the standard solutions similar?

2. Do Zeno's paradoxes point to a serious worry about space?

3. Can we solve the paradoxes without denying the existence of change?

[1]Adapted from http://www.iep.utm.edu/zeno-par/#H3