THINKING WITH MATHEMATICS

An Introduction to Transfinite Mathematics

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Preface

This booklet deals, in essence, with a branch of mathematics which may be unfamiliar to many readers. Some of the ideas contained herein are fairly sophisticated and will require careful reading as well as a considerable amount of thought. On the other hand, the treatment is deliberately informal, and the explanations are presented with an abundance of detail and in nontechnical language. Anyone, therefore, with a lively curiosity and a good background in elementary high school algebra should be able to cope with the material quite effectively.

To most of us at one time or another there must have come recurrent, perplexing thoughts about the nature of something called infinity. What exactly is meant by the term? Is it something to run away from—or at best to write off as undefinable? Or is it something which can be scientifically examined and completely explained by mathematical methods?

Perhaps the truth lies somewhere between the two extremes. At any rate, mathematicians and philosophers have been tussling with questions relating to infinity for a considerable span of time. As a result there has sprung into existence a strange new species of objects called transfinite numbers. What these are, how they behave, and how they can be compared one to another will be the principal subject of this study. When you have finished you may still want to run away from infinity. But at least you will know more about what you're running from.

The theory of transfinite numbers, largely the brain child of the great mathematical genius Georg Cantor, has had a profound influence on both the mathematical and the philosophical thought of our day. Cantor's development of the theory of cardinal and ordinal numbers is a work of great sophistication and bold, penetrating insight.

It will not be possible in this small book to encompass the scope of Cantor's work nor that of his followers in the field. Moreover, it has not seemed desirable to strive for the high level of rigor and logical exactness which the theory is capable of exhibiting.

On the other hand, it is hoped that this brief introduction with its informal glimpses into the surprising "outer reaches" of infinity will start the ball rolling toward further explorations. Much interesting country lies ahead.

Hanover, Indiana JOHN E. YARNELLE

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The number of grains of sand on the beach at Coney Island is much less than a googol —
10,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,-
000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,-
000,000,000,000,000,000. EDWARD KASNER

I could be bounded in a nutshell and count myself a king of infinite space. SHAKESPEARE

The infinite! No other question has ever moved so profoundly the spirit of man. DAVID HILBERT

CHAPTER I

CARDINAL NUMBERS

1. THE LARGEST NUMBER

Three very youthful but nonetheless earnest mathematicians were engaged in a heated discussion, attempting to come to grips with a very perplexing problem.

"How big," the first one wanted to know, "is the biggest number there is?"

"Anybody knows that," said the second member, who seemed to be a kind of self-appointed committee chairman. "It's a zillion!"

"What's a zillion?" persisted the first speaker.

"A zillion's a million million!" the chairman replied with the air of one delivering the coup de grace.

At this point the number three disputant, who had heretofore maintained a discreet silence, came forth with the following query:

"What about a zillion plus one?"

* * *

As it has to many a serious seeker after truth since the time when man first began to probe into the shadowy, recondite mysteries of number, this question proved to be a real "stumper"!

There was, in fact, a sustained period of silence, after which the chairman again picked up the ball—but this time in the manner of one who is manifestly less sure of his ground.

"All right, what about a zillion zillion?"

"You take a zillion zillion," said number three, "and I'll take a zillion zillion plus one!"

Whereupon the chairman opened his mouth as if to make a telling reply, then bit back his words. For he was a fair-minded, thoughtful boy, who undoubtedly became in later years one of the nation's outstanding scientists!

Following a few moments of soul-searching reflection, he made this remarkable utterance:

"I guess, then, there isn't any biggest number."

"Why not?" said the first boy.

"Because no matter how big a number you have, you can always add one more."

A profound observation, and one which can well serve to keynote this introduction to a fascinating area of mathematics, namely, the study of the *infinite*, or, more precisely, the study of *transfinite numbers*.

2. SETS

In putting the seal of approval on the final comment of our youthful chairman it might be well to adjust the language a bit. Suppose we replace the word number by positive integer, the *set* of positive integers (sometimes called *counting numbers*) being the set

$$\{1, 2, 3, 4, \ldots\}.$$

The statement, "There is no biggest number," in more formal dress becomes, "The set of positive integers has no largest element." Note here, in particular, the word *set*. Sets are to play a major role in this story. Hence it seems appropriate at this point to comment briefly on the general theory of sets, taking the inevitable risk that much of what immediately follows may be "old hat" to some of our readers.

We regard a set as merely a collection of objects, any objects at all, not necessarily numbers. This is a very flexible concept. It is also quite basic. The only general restriction placed on the elements of any specified set is that they have some identifying property which makes completely clear the fact of their belonging to this particular set. In other words, given a set S and a general class of objects, we must always be able to tell whether or not a selected object belongs to the set S.

For example, suppose the set S is assumed to be all human beings presently living in Boston. Then, if one chooses any human being at random, there should be no uncertainty: it is either an element in the set S, or it isn't. As a simpler example, let S be the set of all numbers larger than 100. Acting on this assumption, we can confidently state that 105 belongs to S, 50 does not.

The general theory of sets is one of compelling interest and importance. It can, in fact, be regarded as the foundation stone upon

which all of mathematics may be erected. The reader is therefore strongly urged to delve deeper into this topic which has been called "one of the boldest creations of the human mind."

For the purposes of the present study we shall confine ourselves to only a few aspects of the subject, generating thereby, we hope, an insatiable appetite for more.

3. CARDINAL NUMBERS

Our chief concern, as indicated above, is with so-called transfinite numbers. We shall, to put it somewhat more glamorously, be "exploring the infinite." What, then, has this to do with the theory of sets, or, vice versa, what do sets have to do with infinity? As we shall shortly be led to observe, a great deal!

Let's begin by considering one specific property which we shall call the *cardinal number* (or cardinality) of a set.

Initially this property may seem rather trivial and scarcely worth making a major production over. The plot, however, thickens as we move along. Loosely speaking, the cardinal number of a set is the number of elements it contains. A set consisting of a baseball team (not necessarily from St. Louis) would have the cardinal number nine. If a set is defined as the vertices of a square, its cardinality is four. A twelve-volume set of books has cardinality twelve. Accordingly, two sets with the same number of elements have the same cardinality.

As indicated above, there doesn't seem to be very much to this idea at first. If one can actually *count* the number of elements in a set, or arrive at this number through some other avenue of knowledge, as in the case, for example, of the set of all persons in the United States according to the 1960 Census, then the cardinal number is relatively easy to determine.

But what if it is *not* possible to count *all* of the elements? Can one, for instance, define the cardinal number of a set whose elements cannot all be counted? If so, would all such sets have the *same* cardinal number? These and other questions like them lie at the very heart of the matter. We shall seek to provide some, at least, of the answers.

4. COUNTING

Before making a sortie into this relatively unknown and possibly dangerous new territory, let's examine more carefully what we actually mean by *counting*. Everyone knows in general what the word signifies. Hardly a day goes by without our having to engage in the process in one way or another. Even at night we occasionally count sheep!

What is needed, however, is a slightly more precise notion which can be phrased in mathematical terms in such a way as to provide a means for forming ideas about infinity.

Let's begin with a general set of, say, 10 elements. Suppose, then, that these elements are arranged in a certain way and each element labeled according to this particular arrangement as follows:

$$\{s_1, s_2, s_3, s_4, \cdots, s_{10}\}$$ [1]

If we then consider the set of positive integers from 1 to 10 inclusive, i.e.,

$$\{1, 2, 3, 4, \cdots, 10\},$$

it should be clear that we can match elements in our general set with elements in the integer set so that each general element corresponds to exactly one integer, and vice versa. This type of pairing is an important mathematical phenomenon called (quite understandably) a *one-to-one* correspondence. In the above instance the correspondence is readily achieved by matching each subscript with its "fellow integer."

It may be interesting to observe that we do form a one-to-one correspondence of this sort, though perhaps unconsciously, whenever we go through the ordinary business of counting. We look, for example, at a crowd of people in a room. Wanting to know how many there are, we first work out some kind of ordering process, often starting at the back (or front) of the room and attempting to form, if possible, a pattern of rows. Then, beginning with the first (or last) row and (if we happen to be a product of Western culture) proceeding from left to right, we say:

"One, two, three, four, et cetera"

But this is precisely the act of forming a one-to-one correspondence between, on the one hand, a set of people and, on the other, a set of positive integers.

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[1] The conventional technique for listing the elements of a set is to enclose them in braces, { }. Throughout this study we shall, in general, use capital letters to denote sets and small letters to denote elements of sets.

Suppose we are counting the cards in a bridge deck. This is again the formation of a one-to-one correspondence in which the top card "corresponds" to the integer 1, the second from the top corresponds to the integer 2, and so forth.

With this idea in mind we are now in a position to construct a more precise definition of what is meant by the cardinal number of a set.

Definition: *A set S is said to have cardinal number n if and only if the elements of S can be put into a one-to-one correspondence with the set of integers*

$$\{1, 2, 3, \cdots, n\}.$$ [1]

Here n is assumed to be any positive integer. The symbol $\{1, 2, 3, \ldots, n\}$ denotes the set of *all* positive integers from 1, in ascending order, up to and including n. We shall use the notation $N(S) = n$ to mean "The cardinal number of the set S is n."

Sets that have the same cardinal number are said to be *equivalent*. It follows that:

Two sets are equivalent if and only if their elements can be put in 1–1 *correspondence with each other.*

Thus it is evident that the concepts of cardinality, equivalence, and 1–1 correspondence are intimately associated one with another.

You have probably at one time or another encountered comments about the so-called *empty set*, or set devoid of elements, usually signified by the symbol $\emptyset$. This set has considerable significance in the general theory. We shall not, however, in this discussion consider the cardinality of $\emptyset$ as zero, though this is sometimes done. In assigning a cardinal number to any set we shall always assume that the set in question is non-empty.

5. INFINITY

We can now form a connecting link between the concept of cardinal number and the general notion of infinity. Though the words finite and infinite have a certain popular meaning, we shall pin things down a bit by means of the following definition.

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[1] A question may arise in the minds of skeptical readers (and these are the best!) as to the possibility of arriving at two different cardinal numbers for the same set by merely rearranging the order. It can be proved, however, that such is not the case. For a discussion of this proof, see Chapter XII of *A Survey of Modern Algebra* by Birkhoff and MacLane. Macmillan, 1953.

Definition: *A set S is finite if and only if its cardinal number is a positive integer. A non-empty set which is not finite is called infinite.*[1]

Though finite sets have their established place in the general scheme of things, it is on the infinite sets that we shall direct our most penetrating gaze. As indicated earlier, the major questions compelling our attention are: 1) Can these sets be assigned a cardinal number? 2) Would the cardinal number of all infinite sets be the same?

The second question, by far the more intriguing, is equivalent to inquiring whether or not there are different kinds of infinity. No doubt to many people there is an "obvious" reply to this: "How could there be different kinds! Infinity is infinity!"

Those, on the other hand, who are privileged to read through these pages may have cause to disagree!

6. SUBSETS

Before we can deal with the question of comparative sizes of infinite sets, we shall need to sharpen up a few notions about comparing sets in general. To do this we shall lean heavily on the concept of *subset.* Many readers are probably familiar with this idea, but perhaps a formal definition may not be amiss as a refresher.

We say that

Definition: *A set S is a subset of a set T if every element in S is also an element in T.*

In symbols, "S is a subset of T" is written $S \subseteq T$.

In general, the notion is a simple one. The set of vowels is clearly a subset of the set of all letters in the alphabet. The set of "even" positive integers is a subset of the set of all positive integers. The set of all blondes is a subset of the set of all human beings.

A slightly more subtle question arises, however, which we shall need to deal with, and that is the question of whether or not a set is a subset of itself.

Study the definition! Now ask yourself, "Is every element of S an element of S?"

[1] A word of caution! There may be a temptation to regard as infinite all sets for which an exact count cannot be made in a literal or physical sense. For example, the set of grains of sand on a beach might seem to be infinite. However, such a set is *finite* even though one might never be able to determine its precise cardinal number. If by a prodigious feat of engineering one were gradually to remove all of the sand from a beach, one would ultimately come to the *last* grain. Not so with infinite sets, as we shall see!

As a further consideration we define a *proper subset* as follows:

Definition: *If S is a subset of T and there exists at least one element of T which is not in S, then S is a proper subset of T.*

For this situation we write $S \subset T$.

Armed with these ideas we can begin our attack on the comparison of sets by establishing a connection between cardinal numbers and the subset concept. For the finite sets such a connection is fairly obvious. The following statements can be formally proved using mathematical induction. We shall grant them intuitive acceptance.

Let a set S have cardinal number m and let a set T have cardinal number n. Then $m \leq n$ if and only if there is a 1–1 correspondence between S and a subset of T. (Here and in the future the symbol 1–1 designates "one-to-one.")

Though we have encountered so far only cardinal numbers which are positive integers, the above statement can and will apply to all cardinalities, including the *transfinite.*

If we were to confine ourselves to a consideration of finite sets, a stronger statement would also hold:

Let m and n be positive integers, and let $N(S) = m$ and $N(T) = n$. If there exists a 1–1 correspondence between S and a proper subset of T, then $m < n$.

On the other hand, we shall soon see (and perchance marvel at the fact) that the latter situation does *not* hold with infinite sets. For these one needs additional evidence to determine whether $m < n$ in the infinite case. But of this, more later.

Meanwhile, as an opening thrust, let us reexamine an old friend, the set of all positive integers, i.e.,

$$\{1, 2, 3, \cdots\}.$$

We have already agreed that such a set has no largest element. Can we now assert that

"The set of positive integers is infinite"?

The answer is almost obvious, to be sure! The assertion, however, does need a bit of verification in the light of our definition. This verification involves a procedure we shall find useful later on.

To show that the set of positive integers, call it P, is infinite, we need merely establish that its cardinal number cannot be a positive integer. Assume, then, for the sake of argument that P *is* finite, i.e., that its cardinal number is a positive integer n. By definition this implies a 1–1 correspondence between P and a set of positive integers $\{1, 2, 3, \ldots, n\}$. Now suppose we denote the first n integers in the set P as $\{p_1, p_2, p_3, \ldots, p_n\}$ where $p_1 = 1, p_2 = 2$, etc. Certainly the 1–1 correspondence between this set and the set $\{1, 2, 3, \ldots, n\}$ is obvious.

However, no matter what p_n is, the integer $p_{n+1} = n + 1$ is also an element in P, as our young friends discovered. But the element p_{n+1} is not in the set $\{p_1, p_2, p_3, \ldots, p_n\}$. Hence we have a 1–1 correspondence between $\{1, 2, 3, \ldots, n\}$ and a *proper* subset of P. Thus if P has a finite cardinal number, this number must be greater than n. But n was assumed to be any positive integer whatsoever! What, then, does this compel us to conclude with respect to the set P?

CHAPTER II

COUNTABLE INFINITIES

1. THE CARDINAL NUMBER $\aleph_0$

Since the set of positive integers is infinite, it cannot have a finite cardinal number. We shall therefore be obliged to assign to it a cardinal number which is not an integer. This necessitates a new symbol. Though a number of candidates have been suggested for the job, the most widely used is the symbol $\aleph_0$, called *aleph zero.*[1]

Thus $N(P) = \aleph_0$, and we have made a brave start in answering the first question in regard to the cardinal number of infinite sets. The answer may strike one as being somewhat arbitrary, to be sure, but this is not untypical in the realm of mathematics and mathematicians, where one frequently encounters the following type pronouncement: "You say there is no such number! Behold, I shall create one!" (Modest fellows, these!)

Once the cardinal number of the set of all positive integers has been designated, it is reasonable to come up with the following:

Definition: *A set S has cardinal number* $\aleph_0$ *if and only if the elements of S can be put into a* 1–1 *correspondence with the set of positive integers* $\{1, 2, 3, \cdots\}$.

Do you see that the above is entirely consistent with an earlier statement about finite sets?

We might, then, consider the set P and all others having the same cardinal number to be "Charter Members" in the "Infinity Club." Such sets are commonly called *countable*,[2] a surprising name since

[1] This symbol is sometimes called *aleph null* or *aleph naught*.

[2] Other synonyms for *countable* which are frequently used in mathematical literature are *enumerable* and *denumerable*.

one is not able actually to "count" *all* of the elements. However, the name does make sense in the following context.

If a set S is to have cardinal number $\aleph_0$, then the assumed one-to-one correspondence with P implies that the elements of S may be labeled in a manner such as

$$\{s_1, s_2, s_3, \cdots, s_n, \cdots\}$$

where every element s_n has exactly one "mate," the number n, among the positive integers, and every positive integer k has exactly one partner, s_k, among the elements of S.

In this sense there is an implicit "counting" process even though it is never *completed*.

We may look upon $\aleph_0$, then, as our first acquaintance among the *transfinite* numbers. Is it the only transfinite number? If not, is it the largest? or possibly the smallest? We shall, in due course, find all this out! In the process of determination there will come to light some strange and wonderful discoveries.

2. HOW SMALL IS $\aleph_0$?

We shall now embark upon the search for other transfinite numbers which, while still not finite, may be smaller than $\aleph_0$. How would one go about this? Recalling that the idea of "smaller" cardinal number has been associated with the notion of subset and, in particular, proper subset, we may have a clue. Let's examine a subset of P which, though proper, is still infinite. Perhaps the simplest one to come to mind is the set S of all positive integers except the number 1. This is a proper subset. It can also be shown to be infinite. (One may bypass the formalities in this connection, noting merely that the elements of S cannot be matched 1–1 with a finite set of integers.)

Suppose, then, we label the elements of S as

$$\{s_1, s_2, s_3, \cdots, s_n, \cdots\}$$

where $s_1 = 2$, $s_2 = 3$, $s_3 = 4$, and in general $s_n = n + 1$. The question is, "Can we match each element of S with exactly one element of P, and vice versa?" How about letting subscripts of each s pair up with corresponding integers of P?

Since this notion of the existence of a one-to-one correspondence is basic to our entire discussion, it might be well, perhaps, to "labor" the point a bit more before we reach a final conclusion. Because we are dealing with infinite sets, we cannot exhibit the correspondence

in toto. What, then, can be used as a clinching argument? Evidently we must forge some sort of connecting link between a general element of S, the set being tested, and a corresponding integer of P which would enable one to always make the association. Since the correspondence must be 1–1, the rule must work in both directions. That is, given any element of S we should know which integer in P to match it with, and given any element of P we should know which element of S must correspond to it.

A convenient way of doing this is as follows: If the set S to be examined is written in the form

$$S = \{s_1, s_2, s_3, \cdots, s_n, \cdots\},$$

then a 1–1 correspondence between S and P is implicit. (Match subscript with integer!) Therefore the task at hand is to devise a scheme for labeling the s's. Given, in other words, a set S, what element are we going to call s_1, what element s_2, and in particular what element will be called s_n? This last is the key to the matter since it furnishes the general rule of association.

For the above example the job has already been done. The given set S was the set of all positive integers from 2 on. We let $s_1 = 2$, $s_2 = 3$, $s_n = n + 1$. Clearly, then, the element 2 in S is to be labeled s_1. It will therefore correspond to the *integer 1* in P. The element 7 in S will be labeled s_6, which associates it with the integer 6 in P.

It should now be evident (perhaps by this time painfully so) that there is then a 1–1 correspondence between S and P. This assures us of the fact that the set S has the cardinal number $\aleph_0$.

But hold on! The set S is a *proper* subset of P. Should it not, then, have a "smaller" cardinal number? Have we run into a contradiction? The answer is "No," since the connection between *proper* subset and "smaller" cardinal number was established for finite sets only. Like the "Twilight Zone," we might say that in the mysterious realm of infinity almost anything can happen. We have seen a 1–1 correspondence between the set P and a proper subset of itself.

Suppose for a set S we choose the positive integers with the first ten left out, i.e., let

$$S = \{11, 12, 13, \cdots\}.$$

By letting $s_1 = 11$, $s_2 = 12$, and $s_n = n + 10$, we have a 1–1 correspondence between P and an apparently even smaller subset. The correspondence, however, tells us that the two sets have the same cardinal number $\aleph_0$.

The reader can readily discern that the choice of how many initial numbers to omit does not materially alter the situation. One can let S be the set of positive integers with the first thousand left out, and still come up with a similar result. Or, for that matter, leave out the first "zillion," or any *finite* number.

Let's consider now a different type of subset. Assume that S is the set of all even numbers beginning with 2, i.e.,

$$S = \{2, 4, 6, \cdots\}.$$

If we let $s_1 = 2$, $s_2 = 4$, and in general $s_n = 2n$, it should again be clear that a 1–1 correspondence exists between the set of positive integers and the set of even positive integers. The surprising thing about this is that the latter, superficially at least, would appear to have only half as many elements as the former, and yet they have the *same* cardinal number $\aleph_0$.

Once again we are witness to the unconventional behavior of infinite sets and of transfinite numbers. The foregoing examples inevitably lead us to the conclusion that a certain infinite set can be put into 1–1 correspondence with a proper subset of itself. As a matter of fact, this turns out to be a characteristic property of infinite sets. The great mathematician Richard Dedekind is credited with the assertion that:

A set is infinite if and only if *it can be put into a* 1–1 *correspondence with a* proper *subset of itself.*

We have already seen examples of this with respect to the set of positive integers. It should also be clear to the reader that the same is true for all sets with cardinal number $\aleph_0$, that is, for all sets which are countably infinite. To affirm the latter we need only realize that any pattern of correspondence between the set P and subsets of itself can be duplicated by all other sets which are countable. Take, for example, the very general countable set

$$S = \{s_1, s_2, s_3, s_4, \cdots, s_n, \cdots\}.$$

We can form a 1–1 correspondence between S and the proper subset

$$\{s_2, s_4, s_6, \cdots\}$$

which perfectly imitates the correspondence between P and the even positive integers.

We also know that the property under consideration does not apply to finite sets. This leaves one question to be settled before we can fully accept the Dedekind definition. The answer will also solve another important problem, namely, the relative "size" of $\aleph_0$.

The key question is this: "Does every infinite set contain a countable subset?" An answer to the question can be formally proved. We shall be content with a very plausible justification. Suppose we let T be any infinite set at all. Furthermore, suppose we begin to list the elements as $\{t_1, t_2, t_3, \cdots\}$. We can continue listing elements without having the list come to an end. (Otherwise T would be finite.) Must there then be a countable subset $\{t_1, t_2, t_3, \cdots, t_n, \cdots\}$?

We have verified the Dedekind assertion. We can also conclude that a countably infinite set, or set with cardinal number $\aleph_0$, is the smallest infinite set.

Perhaps some of you would like to challenge this statement. (And we hope many will do so, at least temporarily.) Let him produce an infinite set, say Q, with the claim that $N(Q) < \aleph_0$. To which we reply that since Q is infinite, it has a countable subset R where R has cardinal number $\aleph_0$. But since R is a subset of Q, it follows from an earlier statement (see page 7) that

$$N(R) \leq N(Q)$$

which contradicts the claim that Q has a cardinal number smaller than $\aleph_0$.

Aleph zero having been recognized as the "smallest" transfinite cardinal number, the question now virtually demands to be answered, "Are there larger ones about, and if so, how do we find them?"

3. HOW LARGE IS $\aleph_0$?

We must now swing the pendulum over to the other side. In other words, instead of seeking out proper subsets of P, we must look for "larger" sets of which P itself is a proper subset. Let the reader be warned that there are frustrations ahead! In fact, one is led almost to despair of ever finding a "larger" infinity. However, these very frustrations offer much illuminating insight. Also, to the persevering searcher, there is a pot of gold at the end!

As an opening gambit, let's consider the set I of all integers. Certainly this seems to be a larger set, including, as it does, all the negatives of the counting numbers, and zero. One might look upon it as a bit more than twice as large as P. Therefore, since P is a proper subset of I, we can infer that

$$N(P) \leq N(I).$$

But now comes the moment of truth! Suppose we consider the following designations:

$$I = \{i_1, i_2, i_3, i_4, \cdots, i_n, \cdots\}$$

where

$$i_1 = 0, i_2 = -1, i_3 = 1, i_4, = -2, i_5 = 2, \text{ et cetera.}$$

To describe the general scheme we can let

$$i_1 = 0, \quad i_n = -\frac{n}{2} \quad \text{for} \quad n = 2, 4, 6, \cdots$$

and

$$i_n = \frac{n-1}{2} \quad \text{for} \quad n = 3, 5, 7, \cdots.$$

This assures us a 1-1 correspondence between all the elements of I and the positive integers. As a test, suppose we choose, for example, elements 13 and 24 from P. Is it clear that the corresponding elements in I are 6 and -12 respectively? To go the other way we merely need to reverse the formula. For instance, suppose we select the number -5 from the set I and seek its correspondent in P. Since -5 is negative we choose

$$i_n = -\frac{n}{2}$$

$$\text{i.e.,} \qquad -5 = -\frac{n}{2}$$

$$\text{whence} \qquad -10 = -n \qquad \text{and} \qquad 10 = n.$$

Thus for the integer -5 in I the corresponding positive integer in P is 10.

From all of this we are obliged to conclude (with a twinge of disappointment) that the set of all integers is countable. That is,

$$N(I) = \aleph_0.$$

What next? This time we shall take on a vastly larger set (in appearance, that is), namely, the set of all rational numbers. You may recall that this is the set of all numbers which can be expressed as the quotient of two integers, with the customary proviso that division by zero is excluded. This set includes a staggering multitude of fractions in addition to all of the integers previously considered!

For convenience we shall begin with the positive rational numbers. The extension to include all negative numbers and zero will follow shortly.

Suppose we call the set of positive rational numbers R_p, and let us assume a listing which begins in the usual way as

$$R_p = \{r_1, r_2, r_3, \cdots, r_n, \cdots\}.$$

Now we can construct an array of the positive rational numbers, as follows. Along the first row we list, in the usual order, the positive integers:

$$1, 2, 3, 4, \cdots$$

To be consistent with the definition of rational number, we should perhaps consider each of these as having a number 1 denominator, i.e.,

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \text{ et cetera.}$$

In the second row we place all fractions having the denominator 2, in the third row all fractions having the denominator 3, and so forth. The numerators in each case will still be the positive integers in the conventional order.

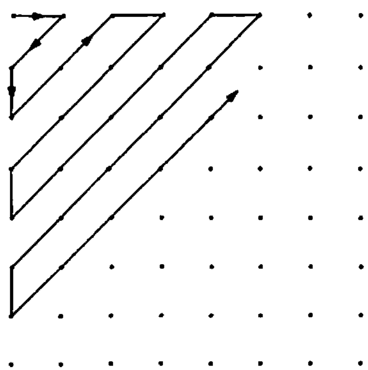
Actually we are talking about an infinite number of rows, each containing an infinite number of elements—a pretty formidable array, and one which could obviously never be completed. It will only be necessary, however, to exhibit a small fragment of the upper left-hand corner to illustrate the general scheme which we are about to describe. The pattern should look somewhat as follows:

$$\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \frac{6}{2} & \cdots & \\
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \cdots & & \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \cdots & & & \\
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \cdots & & & &
\end{array}$$

Even from this small beginning it is evident that some duplication is involved. However, this needn't pose a problem. We may simply assume that all fractions which have not been reduced to lowest terms, i.e., fractions whose numerator and denominator are not relatively prime,[1] are to be eliminated. In the long run, no harm will be done to the great plan.

Now suppose we construct a broken-line path beginning at the number 1 and consisting of horizontals, verticals, and diagonals, as indicated in the sketch below. Note the direction of the line. Note also that elements to be disregarded have been marked with an asterisk.

1	2	3	4	5	6	7	. . .
$\frac{1}{2}$	$\frac{2^*}{2}$	$\frac{3}{2}$	$\frac{4^*}{2}$	$\frac{5}{2}$	$\frac{6^*}{2}$	$\frac{7}{2}$	. . .
$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3^*}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{6^*}{3}$	$\frac{7}{3}$	. . .
$\frac{1}{4}$	$\frac{2^*}{4}$	$\frac{3}{4}$	$\frac{4^*}{4}$	$\frac{5}{4}$	$\frac{6^*}{4}$	$\frac{7}{4}$	. . .
$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5^*}{5}$	$\frac{6}{5}$	$\frac{7}{5}$	. . .
$\frac{1}{6}$	$\frac{2^*}{6}$	$\frac{3^*}{6}$	$\frac{4^*}{6}$	$\frac{5}{6}$	$\frac{6^*}{6}$	$\frac{7}{6}$	. . .
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .



Listing the numbers in the order determined by the broken line (with appropriate omissions), we have $1, 2, \frac{1}{2}, \frac{1}{3}, \frac{2}{2}^*$ (omit), $3, 4, \frac{3}{2}, \cdots$. Thus $r_1 = 1, r_2 = 2, r_3 = \frac{1}{2}, r_4 = \frac{1}{3}, r_5 = 3, r_6 = 4, r_7 = \frac{3}{2}$, and so forth.

From this it should be clear that each positive rational number will appear once and only once in the list, and will have as its correspondent exactly one positive integer. For example, it is evident (after some careful counting) that the rational number $\frac{1}{4}$ corresponds to the positive integer 9, and that the rational number 6 corresponds to the integer 12, i.e., $\frac{1}{4}$ is ninth on the list and 6 is twelfth.

Because of the necessary deletion of unreduced fractions, the problem of devising a formula for r_n which would indicate the general correspondence is a knotty one. It will not be attempted here. For an alternative approach see footnote reference on page 5.

[1] Two numbers are called *relatively prime* (i.e., one is prime to the other) if their only common divisors are 1 and -1. For example: 8 and 25. Neither 8 nor 25 is prime, but they are relatively prime.

The array, however, of numbers and lines should be sufficient evidence to support the conclusion that a 1-1[1] correspondence does in fact exist. For additional affirmation the reader is encouraged to do the following exercises.

EXERCISES. Find the missing partners in the following table:

Positive Rationals		Positive Integers
1	⟷	1
$\frac{1}{4}$	⟷	9
6	⟷	12
$\frac{2}{5}$	⟷	?
?	⟷	20
$\frac{2}{7}$	⟷	?
?	⟷	30
$\frac{3}{8}$	⟷	?

One may go on with further matchings if the game is interesting. For example, what integer corresponds to the rational number $\frac{5}{7}$? What rational number corresponds to the integer 50?

Assured that a 1–1 correspondence does exist between the positive rationals and our basic set P, we need only adjoin the negative rationals and zero to make our task complete.

This can be done without extravagance of detail by merely indicating the following arranged listing. Suppose we call R the set of *all* rationals. Using r_n, as before, to indicate a positive rational number, we can form a counting arrangement as follows:

$$R = \{0, -r_1, r_1, -r_2, r_2, -r_3, r_3, \cdots\}$$

The countability of the set of *all* rationals can thus be derived from the countability of positive rationals in somewhat the same manner as the cardinal number of all integers was derived from that of the positive integers. Thus, it follows, *mirabile dictu!*, that

$$N(R) = \aleph_0.$$

[1] A double arrow (⟷), as in the table, is frequently used to show that the correspondence is 1–1.

4. SUMS AND PRODUCTS OF CARDINAL NUMBERS

The time has now arrived for forming a few conjectures about the inner nature of cardinal numbers and especially about the character of our new-found but peculiar friend $\aleph_0$.

In first presenting the notion of cardinal number as applied to sets, it was pointed out that cardinal numbers of finite sets could be thought of in much the same way as one views the positive integers themselves. That is, a determination of the cardinal number of a given finite set could be made by the simple process of counting the elements.

Up to now, however, we have said nothing concerning the *arithmetic* of cardinal numbers. In particular, we have not confronted the question of what is meant by the *sum* or *product* of two cardinal numbers. We have, it is true, made certain observations concerning the comparative "sizes" of various sets, but have not as yet been very systematic about it.

We shall now remedy the situation by attempting to set up a carefully defined system of arithmetic for cardinal numbers. Any such system must fulfill three major objectives:

1. It must of necessity involve a relationship to sets and to certain established operations on sets.
2. It must, for finite cardinal numbers, be consistent with the conventional arithmetic of positive integers.
3. It must include (and this of course is the feature attraction of the whole show) the *transfinite* cardinal numbers as well.

The word arithmetic has several connotations. It is used, generally, and often misleadingly, to describe the diverse mathematical activities engaged in by young students in the elementary grades. It may also have a strict technical meaning. We use the term here somewhat informally to include primarily three basic concepts with respect to cardinal numbers, namely, for any two cardinal numbers their *sum*, *product*, and *order* (which one is larger than which). We shall also consider at a later time the notion of exponentiation, i.e., taking powers. Other properties will periodically enter the stage in what may be thought of as "supporting" roles.

The reader may or may not be familiar with the idea of what is meant by the *union* (sometimes called the *logical sum*) of two sets. Very briefly, given two sets S and T, the union of these two sets is a third set containing all elements which are in at least one of the sets

S or T. Calling this third set R, we write in symbols

$$S \cup T = R.$$

EXAMPLES

(1) If $S = \{a, b, c, d\}$ and $T = \{a, c, f, g\}$, then $S \cup T = R = \{a, b, c, d, f, g\}$.

(2) If $S = \{a, b, c\}$ and $T = \{d, e\}$, then $S \cup T = R = \{a, b, c, d, e\}$.

The foregoing examples furnish a key to the situation. In the first case the two sets do have elements in common, to wit, a and c. In the second case there are no common elements. With respect to the latter we say that the two sets are *disjoint*.

With this in mind we can readily establish a connection between set union and the sum of cardinal numbers. To begin with, if we look at the first example, we note that $N(S) = 4$ and $N(T) = 4$, but that $N(R)$ in this instance is 6. Obviously in this case the cardinal number of R is *not* the sum of the cardinal numbers of S and T. The reason, which must be equally self-evident, is that S and T contain common elements, i.e., are not disjoint. On the other hand, in the second example we observe that

$$N(S) = 3,\ N(T) = 2 \text{ and that } N(R) = 5.$$

Whence we can now assert that if X has cardinal number m and Y has cardinal number n, where X and Y are *disjoint* sets, then the *sum* of the cardinal numbers (i.e., $m + n$) is equal to the cardinal number of the union $X \cup Y$.

In symbols, if X and Y are disjoint, then

$$N(X) + N(Y) = N(X \cup Y).$$

At this point it should be emphasized that the cardinal number of a set does not in any sense depend on the special nature of the constituent elements but only on how many there are. For instance

$$\{a, b, c\},\quad \{1, 2, 3\},\quad \{\text{red, white, blue}\},\quad \{\text{do, re, mi}\}$$

all have the cardinal number 3. Thus we may from this standpoint regard all of these sets as *mutually equivalent*. We shall want to emphasize this abstract quality of cardinal numbers by regarding, for example, the cardinal number n as the number of elements in any *arbitrary* representative of the general class of all sets having n elements. Accordingly, we have regarded all countable sets, i.e., sets with the cardinal number $\aleph_0$, as mutually equivalent. Thus when we think of $\aleph_0$ as an abstract cardinal number, we would be quite justified in choosing as an "arbitrary representative" the set of all rational numbers or the set of all even positive integers, or the set of all positive integers larger than 1000, to mention a few.

Our explorations will lead us eventually to many more countable sets. There may also arise some amazement over the fact that all of these should be thought of as mutually equivalent.

But first let's take a look at multiplication. Question: "How shall the product of two cardinal numbers be defined?" Recall an earlier example where

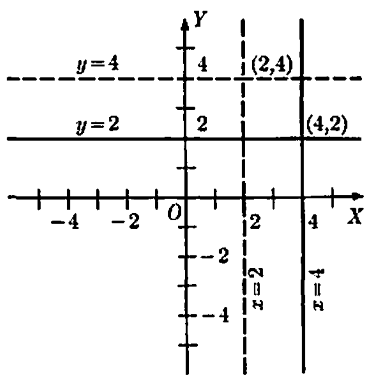
$$S = \{a, b, c\} \quad \text{and} \quad T = \{d, e\}.$$

Suppose one is asked to form all possible pairs, partnerships, committees of two, et cetera, where each pair consists of exactly one element from each set. Such pairs would be

$$(a, d),\ (a, e),\ (b, d),\ (b, e),\ (c, d),\ (c, e).$$

How many pairs in all? The set consisting of all of these pairs is known as the *Cartesian product* of S and T. Note that the elements in the product set are *pairs* and not individual numbers. Symbolically the Cartesian product set is written $S \times T$. To form a Cartesian product it is *not* essential that the sets be disjoint.

More precisely, the Cartesian product is a set of *ordered* pairs, implying that the position within the parentheses is to be considered significant. For example, the pairs (a, c) and (c, a) are treated as dis-



tinct elements. The graph (p. 20) points up such a distinction. The ordered pair (4, 2) is the point of intersection of the lines $x = 4$ and $y = 2$, whereas the ordered pair (2, 4) is the point of intersection of the lines $x = 2$ and $y = 4$.

Returning to the example where $S = \{a, b, c, d\}$ and $T = \{a, c, f, g\}$, each having cardinal number 4, we may form the product $S \times T$ as

$$\{(a, a), (a, c), (a, f), (a, g), (b, a), (b, c), (b, f), (b, g), (c, a), (c, c), (c, f), (c, g), (d, a), (d, c), (d, f), (d, g)\}.$$

How many pairs in this set?

What do these examples suggest? With compelling reasonableness it seems that the product of two cardinal numbers can be based on the Cartesian product of associated sets. We can summarize the foregoing results by means of the following definitions:

Definitions:

(1) *The sum $m + n$ is the cardinal number of any set which is the union of any two disjoint sets having cardinal numbers m and n respectively.*

(2) *The product $m \cdot n$ is the cardinal number of any set which is the Cartesian product of any two sets having cardinal numbers m and n respectively.*

5. THE UNCONVENTIONAL BEHAVIOR OF $\aleph_0$

In establishing criteria for any proposed definition of sums and products we were motivated by the need for preserving the governing regulations for addition and multiplication of positive integers whenever the cardinal numbers under consideration were themselves finite.

Among these are properties with which you are no doubt familiar, namely, the associative and commutative laws for addition and multiplication, and the distributive property of multiplication over addition. Succinctly summarized, these are, respectively, for any 3 cardinal numbers a, b, and c:

$$(a + b) + c = a + (b + c) \quad \text{and} \quad (ab)c = a(bc)$$
$$a + b = b + a \quad \text{and} \quad ab = ba$$
$$a(b + c) = ab + ac, \quad (b + c)a = ba + bc$$

With respect to addition, the commutative and associative properties are carried over from the operation of set union. (Old hat to anyone familiar with the rudiments of set theory and/or Boolean algebra!)

On the matter of multiplication it is evident that the number of possible pairs is independent of the order of selection. For example, if $S = \{x, y, z\}$ and $T = \{v, w\}$, we note that

$$S \times T = \{(x, v), (x, w), (y, v), (y, w), (z, v), (z, w)\}$$

while

$$T \times S = \{(v, x), (v, y), (v, z), (w, x), (w, y), (w, z)\}.$$

Each set of pairs contains 6 elements.

To verify the associative law, you should experiment with three small sets, say $S = \{x, y, z\}$, $T = \{v, w\}$, and $W = \{1, 2\}$. First, form the set of all triples such as

$$\{((x, v), 1), ((y, w), 2), ((z, v), 2), \ldots\}.$$

Then examine a second set of the form

$$\{(x, (v, 1)), (y, (w, 2)), (z, (v, 2)), \cdots\}.$$

A 1–1 correspondence seems to suggest itself. What does this say about the two cardinal numbers?

The conscientious investigator (meaning all of you!) is also urged to verify the distributive property, using the same three sets—or any preferred sets of his own devising.

It should also be interesting to note that for *finite* cardinal numbers the familiar relationship between addition and multiplication can be verified if one employs a special gimmick illustrated by the following example.

Traditionally the concept of multiplication by a positive integer is associated with repeated addition. For instance, a number "times two" is the same as the number "plus" itself. "Set-wise" we can consider an arbitrary set, $S = \{a, b, c\}$, and form the Cartesian product of S and any set with two elements, such as $T = \{1, 2\}$. Clearly,

$$S \times T = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}.$$

If we wish to add the cardinal number of S to itself, we must recognize that addition of cardinal numbers is only defined for disjoint sets. However, we can make our sets disjoint by various means, one of which is to use subscripts. That is, let $S_1 = \{a_1, b_1, c_1\}$ and $S_2 = \{a_2, b_2, c_2\}$. Now the union $S_1 \cup S_2 = \{a_1, b_1, c_1, a_2, b_2, c_2\}$. This is equivalent to the set $S \times T$; i.e., each has 6 elements. Lest the

reader be suspicious of "skulduggery" in the last maneuver, he should recall that the cardinal number of a set is considered to be independent of the particular nature of the constituent elements. In dealing with set S above, we were not concerned with the a, the b, or the c, but merely with the fact of S having 3 elements.

The third and principal criterion for addition and multiplication of cardinal numbers is applicability to transfinite numbers. Since union and Cartesian product are defined for infinite sets we can be assured that this last requirement is satisfied. It now remains to investigate some of the phenomena which result when sums and products are formed involving $\aleph_0$.

Using the same type of reasoning as applied above, we can infer that the associative, commutative, and distributive properties do, in fact, hold. As far as conventional behavior goes, however, this is the end of it.

The various symptoms of unorthodoxy can best be appreciated by reexamining a few of the examples of countably infinite sets which we have already met. As a starter, consider the disjoint sets T and S where $T = \{1, 2, 3, \cdots, 10\}$ and S is the set of all positive integers from 11 on.

For cardinal numbers

$$N(T) = 10 \quad \text{and} \quad N(S) = \aleph_0.$$

Clearly $S \cup T = P$, the set of all positive integers. But $N(P) = \aleph_0$. What does this mean? Is one compelled to conclude that

$$\aleph_0 + 10 = \aleph_0?$$

It is certainly reasonable to extend this idea. Make a few more experiments similar to the above, including a general case with the positive integer n. What is the inevitable conclusion? Can one assert that for any finite cardinal number n

$$\aleph_0 + n = \aleph_0?$$

Going a step farther, suppose we now take a second look at the formation of a 1–1 correspondence between the set of positive integers and the set of *all* integers. In this instance we were adjoining to a set with cardinal number $\aleph_0$ a second set which was also countably infinite, namely, the negative numbers and zero. Clearly the two sets are disjoint. But the set of all integers has cardinal number $\aleph_0$. What does this say about $\aleph_0 + \aleph_0$?

There is no reason to halt the process at this point. Thanks to the associative law we can infer that

$$(\aleph_0 + \aleph_0) + \aleph_0 = \aleph_0$$

and for that matter

$$\aleph_0 + \aleph_0 + \cdots + \aleph_0 = \aleph_0$$

for any *finite* number of terms.

A corollary question straightway arises. Would the relation

$$\aleph_0 + \aleph_0 + \cdots + \aleph_0 = \aleph_0$$

hold for an *infinite* number of terms, in particular a countably infinite number? Let's first see if you can arrive at an answer on the basis of an example. Consider the positive rational numbers and recall the suggested arrangement in rows and columns:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \cdots & \\ \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \cdots & & \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \cdots & & & \end{array}$$

After the unreduced fractions have been discarded we have an infinite number of disjoint sets, i.e. an infinite number of rows. Moreover, each row itself is an infinite set. Thus the totality of elements in the above array might be thought of as the union of a countably infinite number of countably infinite disjoint sets. But we have already established the fact that the positive rational numbers have cardinality $\aleph_0$. Conclusion?

Now we must ask whether the above implies that

$$\aleph_0 \cdot \aleph_0 = \aleph_0.$$

Under the assumption that multiplication is nothing more than reiterated addition, we might reasonably be tempted to answer

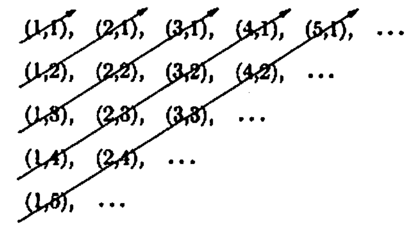
"Yes." However, since this has not been cleared in the case of transfinite numbers (in fact, we do not actually know what it means to "add" an infinite number of quantities), there should still remain a margin of uncertainty. We shall have to test our intuition.

But how? By a reexamination of the definition of the product of two cardinal numbers. Recall that for two cardinal numbers m and n the product $m \cdot n$ is the cardinal number of any set which is the Cartesian product of any two sets having cardinal numbers m and n respectively.

Suppose we first look at a set S with cardinal number $\aleph_0$ and a second set T with cardinal number $\aleph_0$. As the simplest examples let

$$S = \{1, 2, 3, \cdots\} \qquad \text{and} \qquad T = \{1, 2, 3, \cdots\}.$$

The Cartesian product is the following set of pairs:



Arranged as above, they present a picture similar to that of the positive rational numbers, only this time without duplications. Hence no need for discards. The general situation, however, is theoretically the same. Can you set up an enumeration scheme which places all of the above elements in a 1–1 correspondence with the set P of positive integers? For the sake of variety use the diagonal pattern suggested in the illustration.

Good old intuition has again proved a reliable witness. We are now in a position to make a general assertion. This will include the case in which one of the sets may be finite. (You can easily verify the latter. Note that a diagonal-type pattern, such as the one above, does not require an infinite number of rows in order to yield a 1–1 correspondence with the set of positive integers.)

Let n be a cardinal number which is finite. Then

$$n \cdot \aleph_0 = \aleph_0$$

and

$$\aleph_0 \cdot \aleph_0 = \aleph_0.$$

We can take another step. Using the associative property (always a helper in time of need!), we note that

$$(\aleph_0 \cdot \aleph_0) \cdot \aleph_0 = \aleph_0 \cdot \aleph_0 = \aleph_0.$$

A further extension is also possible. Is it not true that

$$\aleph_0 \cdot \aleph_0 \cdot \aleph_0 \cdots \aleph_0 = \aleph_0$$

for any finite number of factors? What about the case in which the number of factors is countably infinite?

A perfectly fair question—and one which certainly deserves an answer. But here lest intuition carry us too far we must definitely raise a warning signal! We have not set up any apparatus as yet for determining what is meant by the product of an infinite number of factors.

To do this we shall have to delve very carefully into the general concept of exponents and exponentiation. In other words, we know the meaning of

$$m^n$$

when m and n are positive integers. But for m and n as cardinal numbers, and, in particular, transfinite cardinal numbers, there may be a new and different story. We shall tell this story in the near future.

For now, however, we must resume the quest for transfinite cardinal numbers other than (i.e., greater than) $\aleph_0$. A quest which up to the present sitting seems to be leading us nowhere!

We have learned that if one adds any finite cardinal number to $\aleph_0$, the result is still $\aleph_0$. What's more, one can even add a countably infinite cardinal number to $\aleph_0$ and still obtain $\aleph_0$. Do this a countably infinite number of times and what do you get? $\aleph_0$. Multiply $\aleph_0$ by itself. Repeat the process again and again, even a thousand or a "zillion" times, and there staring us smugly in the face is still

$$\aleph_0.$$

This probably raises some grave doubts as to whether we can ever get something larger. Nevertheless we shall continue the fight. It may be possible to find a set which, while infinite, is not countably so. If and when such a set is discovered, we can then penetrate more deeply into the general concept of transfinite arithmetic.

6. ALGEBRAIC NUMBERS

To look for bigger game we must certainly go beyond the set of rational numbers, which is the largest set we have thus far examined. Nearly all of you must already have encountered some numbers which cannot be expressed as the quotient of two integers. The square root of 2 (usually written $\sqrt{2}$) is a well-known example. That $\sqrt{2}$ is irrational can be proved. The proof can be found in many good algebra texts.[1]

It can also be proved that other numbers such as $\sqrt{3}$, $\sqrt{5}$, $\sqrt{10}$, $\sqrt[3]{5}$, $\sqrt[5]{3}$, et cetera, are likewise irrational, but we shall not attempt to do so here. If you are a bit rusty on the proof concerning the irrationality of $\sqrt{2}$, for example, we strongly urge some independent research.

We now wish to adjoin all numbers of the above type to the set of rationals, thereby creating a truly vast assemblage. To do this conveniently we'll need to borrow a page from algebra; that is, we want to consider the set of all so-called *real algebraic numbers.*

By a real algebraic number is meant any real root of an equation of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where the a's are all integers with a_n not zero, and where the n's are positive integers. Lest the above notation be a bit bewildering, we offer some specific examples below:

$$3x^2 - 5x + 6 = 0$$
$$x^3 - 7 = 0$$
$$x^4 - 6x^3 + 2x^2 - x + 1 = 0$$

and so forth.

In the present situation we are including only real roots, though similar results can be obtained for all complex roots as well.

It should be evident that the set of real algebraic numbers takes in a tremendous amount of territory. It contains all rational numbers and all real roots of these rational numbers. Surely this must be a likely candidate for an infinite set with cardinal number greater than $\aleph_0$. But let us look into the matter.

[1] See Exercise 12, page 8 of *Elementary Mathematical Analysis* by T. Herberg and J. Bristol. Published by D. C. Heath and Company, Boston, 1962.

In considering equations of the type described, we may assume that the leading coefficient is positive. This is legitimate since any equation may have all of its terms multiplied by (-1) without any consequent change in roots. The left member of the equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

under the indicated restrictions is, as most of you know, a polynomial. With any polynomial of the above type, we can associate a certain measurement called its *height*, which we shall designate by the letter h. This so-called height is a positive integer and is defined as

$$h = n + a_n + |a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0|.$$

Here n is the exponent showing the highest power of x. It is usually called the *degree* of the polynomial. We also use the familiar symbol $|\ \ |$ to indicate *absolute value*. Informally speaking, an absolute value symbol has the effect of changing all negative numbers to the corresponding positive ones and leaving unchanged those which are already positive. Thus, for example,

$$|-7| = 7 \qquad |8| = 8.$$

To clarify the definition of height, a few illustrations are perhaps in order. For the polynomial

$$3x^2 - 5x + 2, \quad h = 2 + 3 + |-5| + |2| = 12.$$

For $\quad x^5 - 8, \quad h = 5 + 1 + |-8| = 14.$

As a bit of practice to crystallize understanding, you might try the following exercises.

EXERCISES. Calculate h for the following polynomials.

1. $x^4 - 3x + 7$
2. $3x^5 + x^2 - x$
3. $x^2 - 6x + 5$
4. $5x^6 - 12$
5. $x^5 + 3x^4 - x^3 + x^2 - 6x - 8$

The significant point to be made here is that for any specified height h there are only a finite number of polynomials having this designated height. For instance, the only polynomials with height 3 are

$$x^2, \quad 2x, \quad x + 1, \quad x - 1, \quad 3.$$

There are five in all. You are invited to find others, if possible!

If we now form equations by setting each of these equal to zero, we discover that there are exactly 3 different real roots, namely, 0, 1, and -1. The coincidence of having exactly 3 roots for polynomials of height 3 is not to be considered significant. The vital point is that the number of roots is finite.

Now see if you can form all possible polynomials for which $h = 4$. Once these are determined, you will observe that the only *new* real roots contributed are -2, $-\frac{1}{2}$, $\frac{1}{2}$, and 2.

As a further exercise, find all the real roots associated with all possible polynomials of height 5. You should, if all goes well, come up with 12 new roots, that is, 12 roots not already obtained from heights 3 and 4. Don't be discouraged if you can't find all 12 immediately. The equations may be a little tricky! You might also wish to verify the fact that height 2 yields only the root zero.

We are about ready for the punch line. To each height there corresponds a finite number of polynomials, hence a finite number of algebraic numbers. (We assume the reader is familiar with the important algebraic theorem which says, in effect, that no polynomial equation of degree n can have more than n distinct roots.) Accordingly, we can begin our customary listing for the algebraic numbers

$$\{s_1, s_2, s_3, s_4, \cdots, s_n, \cdots\}$$

where the sequence of s's is formed as follows:

We start with the roots of polynomials of height 2, which means that $s_1 = 0$. Then, in order of magnitude, we continue with the new roots contributed by height 3, then by height 4, and so on. The sequence, in line with previous findings, would begin to shape up as $s_2 = -1$, $s_3 = 1$, $s_4 = -2$, $s_5 = -\frac{1}{2}$, $s_6 = \frac{1}{2}$, $s_7 = 2$, $s_8 = -3$, $s_9 = -\frac{1}{2} - \frac{1}{2}\sqrt{5}$.

Note that the last two entries have not been previously exhibited. These can be used as a helpful hint for "operation $h = 5$."

This sequential listing, even though it represents only the barest beginning, should set in motion an inescapable train of thought. If we continue to list only those roots which have not appeared before as we progress upward to new heights along the scale of positive integers, we shall have a sequence of distinct algebraic numbers. Furthermore, since *every* polynomial must have some positive integer as height, this procedure guarantees that every algebraic number must eventually be included.

Is it now evident that such a listing does, in fact, delineate a 1–1 correspondence between the algebraic numbers and the positive integers? In other words, is this whopping new set no more than countably infinite? If so, then it too has cardinal number $\aleph_0$.

The inclusion of this particular example of a "mammoth" set with cardinal number $\aleph_0$ is not intended merely to discourage you from further exploration. It will be useful in forming a conclusion about another surprising set, the set of so-called *transcendental numbers*, which we shall meet later on.

CHAPTER III

UNCOUNTABLE INFINITIES

1. THE UNIT INTERVAL

Our chapter title has undoubtedly given the show away. But the outcome is one which you must have long suspected, both because of the psychological buildup and also in the light of your own excellent intuition: there must be cardinal numbers greater than $\aleph_0$. Else why all the to-do about transfinite arithmetic in the first place?

Up to now our readers must have had numerous sensations similar to those of a movie-goer who, witnessing a familiar scene, says to his companion, "This is where we came in!"

Though we have kept encountering "newer and bigger" sets, there always seems to be an ingenious way of establishing a 1–1 correspondence with the positive integers. There is definite justification at this stage for forming a conjecture that, given sufficient ingenuity, perseverance, and time, one could do this for all sets. Perhaps a more disquieting notion is the following: Even if one couldn't actually find a way, how would one ever be able to prove that such a thing could *not* be accomplished?

This is a very significant question in mathematics and one which has given rise to some of the most highly creative work in the history of thought, namely, the establishment of definite proof that a certain type of problem could *not* have a solution, or that a certain line of research could *not* produce fruitful results.

In this connection one often hears attempted arguments on the other side, such as "They said man could never fly!" or "They said we could never reach the moon!" The inference being that nothing is really impossible.

Such assertions to the contrary notwithstanding, there have been many elegant and valid proofs attesting to the non-existence of solutions. Such proofs often require a higher level of sophistication than those which merely exhibit an actual answer. We are about to attempt such a feat. Hence you may regard the foregoing comments as a friendly warning to be on guard. There may be some heavy weather ahead!

From the start we shall have to assume a basic familiarity on your part with the so-called system of *real numbers*. The set which we shall be examining to begin with is, in fact, the set of all real numbers between 0 and 1. We shall wish to include the number 1 itself but not zero. In this discussion we shall accept, without argument or fuss, the existence of a 1–1 correspondence between real numbers and points on a line. This agreement will enable us, when desirable, to interchange the above set of real numbers with the set of points in the *half-open interval* (0, 1], the bracket indicating the inclusion of 1, the parenthesis indicating the exclusion of zero. In examining this set we shall not presume on our reader's part a knowledge of all there is to know about real numbers. We shall merely agree, and ask you to do so likewise, that every number in this set can be written as a nonterminating decimal

$$.a_1a_2a_3a_4\cdots$$

where the a's may be thought of as any of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 subject to a single restriction which will be indicated shortly.

No doubt there is cause for concern on the part of some over the statement that *every* real number between 0 and 1 may be written as a *nonterminating* decimal. Many of us have been brought up in the fine old tradition that the rational numbers are distinguishable from the irrationals by means of their respective decimal expansions. We have been told that for rationals the expansion takes the form of either a repeating or a *terminating* decimal. For example, $\frac{1}{3} = .333\cdots$ while $\frac{1}{4} = .25$.

To overcome this obstacle and at the same time to guarantee that each number can be expressed in a *unique* manner, we shall stipulate that all so-called terminating decimals be expressed with a nonterminating succession of 9's. Thus $\frac{1}{2}$ is to be written as $.4999\cdots$, $\frac{1}{5}$ as $.1999\cdots$, $\frac{3}{4}$ as $.74999\cdots$, 1 as $.999\cdots$, and so forth.

Assume, then, that each real number in the prescribed set is written decimally. Furthermore, for the sake of argument, let us first make the assumption that the set is countable, i.e., that we have some sort of ordered listing or counting of elements, in our customary pattern,

as follows:

$$\begin{aligned} s_1 &= .a_{11}a_{12}a_{13}a_{14}\cdots \\ s_2 &= .a_{21}a_{22}a_{23}a_{24}\cdots \\ s_3 &= .a_{31}a_{32}a_{33}a_{34}\cdots \\ s_4 &= .a_{41}a_{42}a_{43}a_{44}\cdots \\ &\cdots\cdots\cdots\cdots\cdots \\ s_n &= .a_{n1}a_{n2}a_{n3}a_{n4}\cdots \\ &\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

For those accustomed to *matrix* notation, the double subscripts will be familiar. To the rest, perhaps a word of explanation is due. The symbol a_{23}, for example, is to be regarded as having two subscripts, 2 and 3 (not the single number twenty-three). The left subscript 2 shows that this digit belongs to the second real number (subscript 2 relating it to s_2). The right subscript 3 locates the digit in the *third* decimal place. Accordingly, a_{57} would represent the digit in the seventh decimal place of the fifth real number; a_{66}, the digit in the sixth decimal place of the sixth real number.

Now that the notation is clear, we can examine our assumption more in detail. We are saying, in fact, that a correspondence can be thought of as existing between each integer n and a nonterminating decimal of the form

$$s_n = .a_{n1}a_{n2}a_{n3}a_{n4}\cdots.$$

This means that one would be theoretically able to count the real numbers according to some plan. Note specifically that we are not saying what the plan is. Nor are we attempting to identify any particular element. We do not say, for example, that

$$s_7 = .352178314092\cdots$$

and

$$s_8 = .2598461024\cdots.$$

We are merely supposing that such an arrangement *is* possible and that it will include all of the real numbers under consideration. Here, then, is the hypothetical setup:

We have, as before, a countably infinite number of rows, each row consisting of a decimal point followed by a countably infinite number of digits. As stipulated, each row must be different in at least some way from every other. The difference can be slight. It might occur in only one decimal place. For example,

$$.4357298502649 \cdots$$

and

$$.4357298512649 \text{ " " "}$$

are different real numbers, though it might take a careful eye to detect the difference. Both would have to be included in any complete listing.

Now recall that the second part of the basic assumption was that all real numbers in the interval (0, 1] were to be included in the listing. We propose, then, to demonstrate that no matter what the arrangement, no matter how the listing is conceived, there will always be at least one real number belonging in the stated interval which has been inevitably left out of the listing. A kind of tricky business, to be sure, but follow if you can!

Let any proposed listing be represented as before:

$$\begin{aligned} s_1 &= a_{11}a_{12}a_{13}a_{14}a_{15}\cdots \\ s_2 &= a_{21}a_{22}a_{23}a_{24}a_{25}\cdots \\ s_3 &= a_{31}a_{32}a_{33}a_{34}a_{35}\cdots \\ s_4 &= a_{41}a_{42}a_{43}a_{44}a_{45}\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ s_n &= a_{n1}a_{n2}a_{n3}a_{n4}a_{n5}\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

Now form the decimal

$$.a_{11}a_{22}a_{33}a_{44}\cdots a_{nn}\cdots$$

by selecting from the master list those digits represented with duplicate subscripts. (The selected digits may also be thought of as coming from a diagonal line illustrated by the arrow.) This is clearly a real number in (0, 1]. Hence it should appear somewhere on the list, say in the xth position, as the decimal expansion of the number s_x.

Now with reference to s_x, construct a new decimal

$$.b_{11}b_{22}b_{33}b_{44}b_{55}b_{66}\cdots$$

with the following stipulations: one, each b_{nn} is different from the corresponding a_{nn} in s_x; and two, no b_{nn} is zero.

This new number, since it has no zeros, must be a nonterminating decimal. It is a real number in the prescribed interval. Call it t_x. Now note the following very carefully:

The number t_x is different *in some respect from every number on the list. In other words, the real number t_x has* not *been included in the original list, contrary to the hypothesis which assumed that the list was complete.*

In presenting an argument such as this, one must be prepared to meet a challenge or two. For instance, it might occur to someone to ask, "How do we know it is not on the list since we have no actual list to refer to, nor can such a list be actually exhibited?" We can deal with this challenge by returning the ball as follows:

"Very well then, concede that it is on the list. It must appear somewhere in the counting scale, say as the mth number, or decimal expansion of s_m. But the decimal expansion of s_m must contain a digit in the mth place, this digit being a_{mm}. On the other hand, we know that the mth digit of t_x is b_{mm} and by construction

$$b_{mm} \neq a_{mm}.$$

Hence, though s_m may agree with t_x in every other respect, we *know* that its mth decimal is different. Consequently it is a different real number. And, for similar reasons, it is different from any other s on the proposed list."

A second question might concern the possibility of constructing an alternative original list which would include t_x. But we must recall that the original notations $.a_{11}a_{12}a_{13}\cdots$ were quite arbitrary. They merely stood for *any possible listing*.

2. THE CARDINAL NUMBER C

The foregoing arguments lead to a major assertion:

The set of real numbers, or points, in the interval (0, 1] *is not countable.*

Let us call this set J. We are asserting, in other words, that the set J does not have cardinality $\aleph_0$. We do know that the set is infinite. Now can you construct a 1-1 correspondence between the positive integers and a subset of J? Try letting

$$1 \longleftrightarrow 1,\ 2 \longleftrightarrow \frac{1}{2},\ 3 \longleftrightarrow \frac{1}{3},\ n \longleftrightarrow \frac{1}{n},\ \text{et cetera.}$$

Thus $\aleph_0$ is less than or equal to the cardinal number of J.

Furthermore, since $\aleph_0$ has been shown to be unequal to J's cardinal number, we may take it that the cardinal number of J is *greater* than $\aleph_0$. Following convention, we shall call this new cardinal number **C**. Thus it appears that

$$\aleph_0 < \mathbf{C}.$$

The concept of *equivalence* has already been examined for finite and countably infinite sets. This can readily be extended. That is, any two sets are said to be equivalent if they have the same cardinal number, i.e., if the elements of one set can be put in 1–1 correspondence with the elements of the other.

3. AN EQUIVALENCE THEOREM

Since the ground has been broken for the recognition of infinities of different sizes, it might be well at this point to state explicitly a theorem which deals with the question of equivalence. This will be given without proof. It is a theorem of considerable importance, known variously as the Bernstein or the Bernstein-Schroeder theorem. The proof is complicated, but worth investigating.

Theorem: *Given two sets S and T, if S is equivalent to a subset of T and if T is equivalent to a subset of S, then S is equivalent to T.*

The converse is clearly true.

As an illustration, consider the two sets $S = \{1, 2, 3, 4, \cdots\}$ and $T = \{2, 4, 6, 8, \cdots\}$. We have already observed that the two sets are equivalent. Let us then see if each set can be shown to be equivalent to a subset of the other. To make the picture more vivid, we shall use proper subsets.

First, there is a fairly obvious 1–1 correspondence between T and a proper subset of S. Let the elements in T be matched with their exact counterparts in S. Now for the reverse, suppose we match each number in S with a number in T which is 4 times as large, i.e., let 1 in S correspond to 4 in T, 2 in S be matched with 8 in T, 3 in S with 12 in T, and so forth. Do you see that this furnishes a 1–1 correspondence between S and a proper subset of T?

With the above theorem and its converse we can deal with the matter of relative size by means of the following criteria:

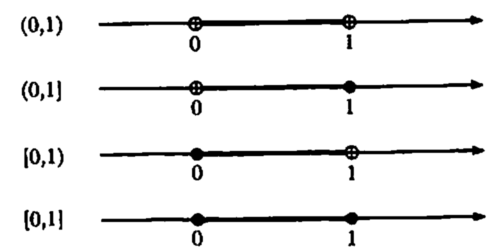
If a set S has cardinal number m and a set T has cardinal number n, then $m < n$ if S is equivalent to a subset of T, but T is not equivalent to a subset of S.

The foregoing conditions were clearly met in the case of $\aleph_0$ and **C**. We have already indicated a 1–1 correspondence between the positive integers and a subset of (0, 1]. The reverse, however, is not possible.

4. OTHER SETS WITH CARDINAL NUMBER C

Having established the fact that the set of all points (real numbers) in the interval (0, 1] has cardinality **C**, we shall now track down other sets with this same cardinal number. Since we will no longer be dealing with countable sets, the business of determining 1–1 correspondences can no longer be accomplished by a simple counting, or enumerating process, as heretofore. We shall need some new machinery. Samples of this will appear shortly.

As a first move, let's look at some other intervals, very much, but not exactly, like the one just considered. These are (0, 1) with neither end point, [0, 1) with 0 but not 1, and finally [0, 1] with both end points in the act. This last interval is often referred to as the *continuum*. (You can detect a reason for choosing the letter **C**.) The four intervals under consideration can be pictured as follows: (The open circle indicates that the point is not included.)



To show that (0, 1] is equivalent to (0, 1), we shall need a special device. For convenience we shall let j stand for any element in (0, 1], our original set, J. Let k be any element in (0, 1); call this set K. The crucial idea here is that j can have the value 1, but k cannot.

Our correspondence scheme will involve a kind of subtraction process. We start with the numbers of J which are larger than $\frac{1}{2}$ and less than or equal to 1. That is, all j such that

$$\tfrac{1}{2} < j \leq 1.$$

Corresponding to any one of these numbers we let $k = \frac{3}{2} - j$. For example, let the point $j = 1$ in J correspond to $k = \frac{3}{2} - 1 = \frac{1}{2}$

in K. Thus $j = \frac{3}{4}$ would correspond to $k = \frac{3}{2} - \frac{3}{4} = \frac{3}{4}$, and so forth. Note that since j does not reach $\frac{1}{2}$, k under this correspondence does not reach $\frac{3}{2} - \frac{1}{2}$, or 1. Do you see, then, why the choice of $k = \frac{3}{2} - j$ was made? So far so good! We have taken care of the values of j in the interval $(\frac{1}{2}, 1]$ by setting up a 1–1 correspondence between these and the values of k in the interval $[\frac{1}{2}, 1)$. We observe, in other words, that by letting $k = \frac{3}{2} - j$, there will be for every j where $\frac{1}{2} < j \leq 1$, exactly one partner k for every k such that $\frac{1}{2} \leq k < 1$. As j goes from 1 down toward $\frac{1}{2}$, k goes from $\frac{1}{2}$ up toward 1.

Now suppose we do the same sort of thing with respect to another pair of intervals. This time split the rest of J in half by considering every point in the interval $(\frac{1}{4}, \frac{1}{2}]$, i.e., every j such that

$$\tfrac{1}{4} < j \leq \tfrac{1}{2}.$$

Note here again j may now have the value $\frac{1}{2}$ (which it couldn't before) but not the value $\frac{1}{4}$. We now wish to set up a 1–1 correspondence between these values of j and the values of k in a new interval. We can split the remainder of K in a similar way. This time use the interval $[\frac{1}{4}, \frac{1}{2})$. As before, k does take on the lower value but *not* the upper.

By this time you should have a pretty good hunch as to how the correspondence can be set up. Suppose we try letting

$$k = \tfrac{3}{4} - j$$

in this case. Thus when

$$j = \tfrac{1}{2},\ k = \tfrac{3}{4} - \tfrac{1}{2} = \tfrac{1}{4}, \text{ and so on.}$$

Why, in this instance, did we choose $\frac{3}{4}$ in the formula $k = \frac{3}{4} - j$? Note again that the new subinterval for J is $(\frac{1}{4}, \frac{1}{2}]$ while for K it is $[\frac{1}{4}, \frac{1}{2})$. We might consider the points of J as starting at $\frac{1}{2}$ and going down to, but never quite reaching, $\frac{1}{4}$. The corresponding points of K will then start at $\frac{1}{4}$ and go up to, but never quite reach, $\frac{1}{2}$. This should give a hint as to why we chose $\frac{3}{4}$.

Having come this far, we can give the next step in more compact form. For the interval

$(\frac{1}{8}, \frac{1}{4}]$ of J

and $[\frac{1}{8}, \frac{1}{4})$ of K,

let $k = \frac{3}{8} - j$.

What is the corresponding value of k for $j = \frac{1}{4}$? Do you see that the number $\frac{3}{8}$ in this formula does the same job as $\frac{3}{4}$ in the previous one? It starts k at the bottom, i.e., $\frac{1}{8}$, when j starts at the top, i.e., $\frac{1}{4}$.

This process can be continued indefinitely. To show how it works, see if you can do the following problems.

EXERCISES

1. Describe the next two pairs of intervals.
2. In each case give the equations which establish the correspondence between the j's and k's.

If one were to go on like this forever (we can't!), a 1–1 correspondence between J and K would ultimately be established completely. We shall not labor the point here. Let's say that we have gathered enough circumstantial evidence to convince ourselves that such a correspondence does exist. What, then, does this tell you about the cardinal number of K?

In looking back over the demonstration, there may still be some puzzlement as to just what we were doing and why. Perhaps you are wondering why the correspondence couldn't have been constructed by using one formula for the entire interval. For example, let $k = 1 - j$ for all values of j. Since j is never zero, then k would never be 1. All right so far! However, the interval $(0, 1]$ for j does allow j to have the value 1. By the formula $k = 1 - j$, this would give k the value 0, which is forbidden. The problem facing us is that K, or $(0, 1)$, is open at both ends while J, or $(0, 1]$, is not. Hence, rather than base our correspondence on a single interval, we have been obliged to use a sequence of half intervals. In each of these both J and K were the same type of interval, i.e., both were half-open even though the respective points not included were on opposite ends.

As in many mathematical situations of this sort, one may not catch the full significance of the maneuver in one quick reading. It is often necessary to reread perhaps several times and then, what is even more important, to think about it for a while. What essentially is the difference between this type of approach and the one used for integers?

In our first encounter with $\aleph_0$ we established a 1–1 correspondence between the set P of positive integers and a proper subset S where S contained every element of P except the number 1 itself. In the argument above we have dealt with a very similar situation where K

is a proper subset of J, in which the number 1 is the only missing element. The procedure, however, seems significantly different. Think about this!

Once we have established the fact that (0, 1] and (0, 1) are equivalent,[1] we can show that [0, 1) is also equivalent to (0, 1). As an exercise try it, noting this time that the missing element in K is 0.

A final step is the establishment of a 1–1 correspondence between [0, 1) and [0, 1] where the latter set includes both 0 and 1.

See if you can make this demonstration. Hint: if one agrees to let 0 correspond to 0, the problem looks a lot like the one we started with!

The result of all this (perhaps a small harvest considering the large-scale planting) is an assurance that the cardinal numbers of four slightly different intervals (one open, two half-open, and one closed) are all the same. That is

$$(0, 1),\ (0, 1],\ [0, 1),\ \text{and}\ [0, 1]$$

all have cardinal number **C**.

5. LONGER INTERVALS

There are some interesting results which follow from this conclusion regarding intervals of any given length. This again brings out some of the more surprising aspects of the study of transfinite numbers and/or infinite sets, although by now we may be prepared for most anything!

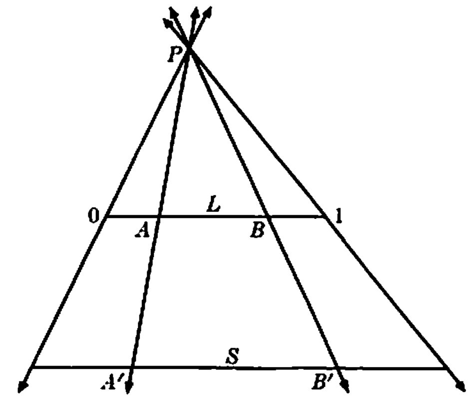
Let us suppose we have an interval from 0 to 1 inclusive, and that this is represented by a line segment L. Now let S be an interval of any finite length. In the figure (p. 41) we shall represent S by a line segment parallel to and drawn beneath the line segment L. Now from a selected point[2] P above L, lines are drawn through the end points of L and S respectively, and also several other lines from P which intersect both L and S, as illustrated.[3]

Without delving too deeply into the geometric implications of this particular exhibit, which is known as a central projection, it should

[1] For convenience we are using interval notation to designate sets of real numbers. Since we have identified, in a sense, the real numbers with points, there should be no confusion.

[2] Select P as the intersection of the lines through the end points of the two segments.

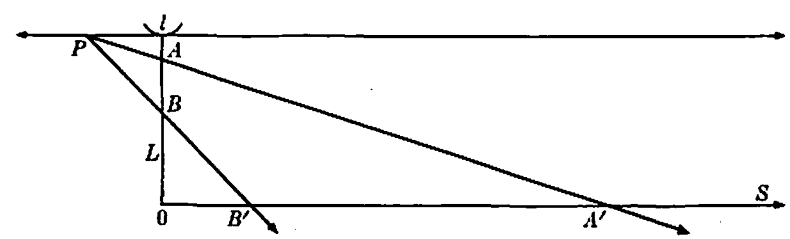
[3] If the length of segment S is less than that of L, then S should be placed between L and P.



be clear from certain premises with respect to points and lines that if lines are drawn from P to S through all points of L, we would have here a 1–1 correspondence. Conclusion?

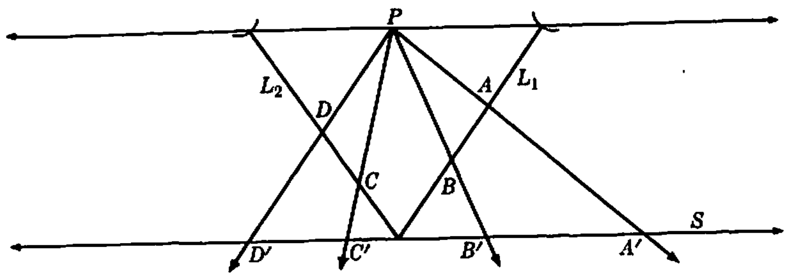
> *The set of all points* (real numbers) *in any finite interval has cardinal number* **C**.

One can go still further and assert that the cardinal number of the set of all points in a *half line* is also **C**. As a suggestion for the demonstration, inspect the following illustration.



Here the curve at the top of line segment L indicates that the upper end point l is not included. The arrow on the right of S implies that the half line S continues indefinitely. Point P is located on a line through the point at l parallel to S. Lines are drawn from P through L and S.

As a final gesture, show that a full line has cardinality **C**. The device indicated below projects a broken, open finite interval onto a *full line.* Here segments L_1 and L_2 are placed so that their open end points lie on a line through P parallel to S. The lengths of L_1 and L_2 are immaterial.



6. MORE ABOUT CARDINAL SUMS

The foregoing considerations give us more ammunition for developing further the arithmetic of transfinite numbers. We have already shown that for any finite cardinal number n

$$\aleph_0 + n = \aleph_0 \quad \text{and} \quad \aleph_0 + \aleph_0 = \aleph_0.$$

It is natural, then, to raise certain questions about $\mathbf{C}$. In particular we wish to find answers to the following:

$$\mathbf{C} + n = ?$$
$$\mathbf{C} + \aleph_0 = ?$$
$$\mathbf{C} + \mathbf{C} = ?$$

Suppose we tackle the last one first.

To begin with, recall that the sum $m + n$ of any two cardinal numbers is the cardinal number of any set which is the union of any two disjoint sets having cardinal numbers m and n respectively. Now let's consider the two disjoint sets S and T where S is the set of all points in the interval $[0, 1]$ and T is the set of all points in the interval $(1, 2]$. Each of these sets has cardinal number $\mathbf{C}$. (You can readily establish this for the second set T by an easily formed 1–1 correspondence between $(0, 1]$ and $(1, 2]$.)

The union of S and T is the set of all points in the interval $[0, 2]$. What is the cardinal number of this set? Remember the central projection scheme in a previous illustration.

Thus the answer to the question $\mathbf{C} + \mathbf{C} = ?$ has now been determined. One concludes inescapably that $\mathbf{C} + \mathbf{C} = \mathbf{C}$.

For the answers to the other two related questions we can use the following idea. Suppose that for any cardinal numbers a and b we are given, that

$$a \leq b$$

and

$$b \leq a.$$

Then b must be equal to a. This is actually a corollary to the Bernstein Theorem (p. 36).

Referring to the question of

$$\mathbf{C} + n$$

where n is any finite cardinal number, we know that $\mathbf{C} + n$ is the cardinal number of any set which is the union of two disjoint sets having $\mathbf{C}$ and n as cardinal numbers. Let T be a set with cardinal number n. There is a 1–1 correspondence between a disjoint set S with cardinal number $\mathbf{C}$ and a subset of $S \cup T$. Hence it follows that

$$\mathbf{C} \leq \mathbf{C} + n.$$

Likewise it can be shown that

$$\mathbf{C} + n \leq \mathbf{C} + \mathbf{C}.$$

The chain is now forged and we have

$$\mathbf{C} \leq \mathbf{C} + n \leq \mathbf{C} + \mathbf{C}.$$

Having just completed the verification that

$$\mathbf{C} = \mathbf{C} + \mathbf{C},$$

we can now make an assertion about $\mathbf{C} + n$.

In like manner one arrives at a conclusion about

$$\mathbf{C} + \aleph_0.$$

Thus ultimately we establish that

$$\mathbf{C} + n = \mathbf{C} + \aleph_0 = \mathbf{C} + \mathbf{C} = \mathbf{C}.$$

There is a consequence of these results which may surprise many of you. In an earlier demonstration we showed that the set of all real algebraic numbers is countable. It can be shown that the set of *all* algebraic numbers including complex roots is also countable. Thus the set of real algebraic numbers is not equivalent to the set of all real numbers There must, then, be some real numbers which are *not* algebraic.

This in itself is not surprising since most of you have already met some of these. For example, $\pi = 3.14159 \cdots$ is one such number. The base of the natural logarithms $e = 2.71828 \cdots$ is another. You might also think of various trigonometric ratios, such as the sine of 25°, as possible candidates. Recall, however, that $\sin 60° = \frac{1}{2}\sqrt{3}$ is algebraic, while $\sin 30° = \frac{1}{2}$ is rational.

A natural question arises concerning the set of *non-algebraic* (often called *transcendental*) numbers. Is it finite? or even countably infinite? If finite, then the real numbers (algebraic plus transcendental) would have cardinal number $\aleph_0 + n$. If countably infinite, the reals would have cardinal number $\aleph_0 + \aleph_0$. How large, then, do you think the set of transcendental numbers must actually be?

7. MORE ABOUT CARDINAL PRODUCTS

If we now turn our attention to multiplication, we can form conclusions about the products $\mathbf{C} \cdot n$, $\mathbf{C} \cdot \aleph_0$, and $\mathbf{C} \cdot \mathbf{C}$.

Initially we must recall again the definition of product with respect to cardinal numbers. If S has cardinal number m and T has cardinal number n, then $m \cdot n$ is the cardinal number of the Cartesian product $S \times T$. Remember, a Cartesian product is the set of all ordered pairs of elements (s, t); s is an element of S, and t is an element of T.

Let's begin the investigation this time by looking at $\aleph_0 \cdot \mathbf{C}$. As defined, the product $\aleph_0 \cdot \mathbf{C}$ is the cardinal number of the Cartesian product of two sets, with cardinalities $\aleph_0$ and $\mathbf{C}$ respectively. For our two sets we may select P, the set of positive integers, and S, the set of all points in the interval $[0, 1)$.

From previous considerations we know that these have the requisite cardinal numbers.

We must examine, then, the set of all pairs (n, s) where n is a positive integer and s is a point in the interval $[0, 1)$, i.e., a real number less than 1 and greater than or equal to 0.

To find the cardinal number of this set of pairs we need to develop a 1–1 correspondence between $P \times S$ and some other set whose cardinality we already know. But what are some of these sets with known cardinal numbers? A quick review tells us that all finite

intervals (of any length) have cardinality $\mathbf{C}$ and that a half line and a full line have cardinality $\mathbf{C}$ also.

How can we use this information to practical advantage in the above case? Suppose we consider the individual elements in our Cartesian product. One subset would be all elements of the form $(1, s)$, where s runs over the set of points in the interval $[0, 1)$. A second subset would contain all elements of the form $(2, s)$, elements in a third subset have the form $(3, s)$, and so forth. This suggests a possible correspondence. What if, for example, we let each element in the first subset correspond to the number (or point) $1 + s$? In other words,

$$(1, 0) \longleftrightarrow 1$$
$$(1, .25) \longleftrightarrow 1.25$$
$$(1, .33) \longleftrightarrow 1.33$$

It should be clear that we would then have a 1–1 correspondence between the elements in our first subset and the points of the half-open interval $[1, 2)$.

What about the second subset? Can this be put in 1–1 correspondence with a second interval? The third subset? Can any subset with elements of the form (n, s), with n a fixed integer, be made to correspond with an interval $[n, n + 1)$? But the totality (union) of all disjoint subsets with elements (n, s) for $n = 1, 2, 3, \cdots$ is actually our Cartesian product, while the totality (union) of all the intervals of the form $[n, n + 1)$ where $n = 1, 2, 3, \cdots$ is a half line with initial point 1. This half line has cardinality $\mathbf{C}$. What, then, is the cardinal number of the Cartesian product? What can be said about $\aleph_0 \cdot \mathbf{C}$?

We can now use the Bernstein corollary encountered before. Without elaboration of detail it should be clear that a set with cardinal number $n \cdot \mathbf{C}$, with n a finite cardinal number, is a Cartesian product with elements of the form (k, s) where k has values $1, 2, 3, \cdots, n$ for a finite integer n. This is clearly a subset of the Cartesian product considered above. Consequently we may conclude that

$$n \cdot \mathbf{C} \leq \aleph_0 \cdot \mathbf{C}.$$

What about the relationship between $\mathbf{C}$ and $n \cdot \mathbf{C}$? Very recently we established a 1–1 correspondence between the points in the interval $[1, 2)$ and the set of all pairs of the form $(1, s)$, i.e., subset number one in the previous demonstration. Thus a set with cardinal number $\mathbf{C}$ is equivalent to a subset of a set with cardinality $n \cdot \mathbf{C}$.

Now it follows that

$$\mathbf{C} \leq n \cdot \mathbf{C} \leq \aleph_0 \cdot \mathbf{C} \quad \text{and} \quad \aleph_0 \cdot \mathbf{C} = \mathbf{C}$$

whence we conclude that $n \cdot \mathbf{C}$ and $\aleph_0 \cdot \mathbf{C}$ are both equal to $\mathbf{C}$.

As Hamlet mournfully inquired, "What ceremony else?" There remains the question of

$$\mathbf{C} \cdot \mathbf{C}.$$

In dealing with this, we shall also uncover another important phenomenon. In other words, the project has a double dividend.

Suppose we begin in a similar manner. Only this time we shall use as our representative set with cardinality $\mathbf{C}$ the set of points (real numbers) in the interval $(0, 1]$. A set with cardinal number $\mathbf{C} \cdot \mathbf{C}$, then, is the set of all pairs (x, y) with x any real number between 0 and 1 excluding zero, and y any real number in the *same* set.

We have already established the fact that each element in either of our two sets can be uniquely expressed as a nonterminating decimal. In the Cartesian product, therefore, any element (x, y) may be thought of as a pair of nonterminating decimals.

The strategy to be followed should now suggest itself. We wish to inquire whether or not a 1–1 correspondence can be set up between our set of pairs and some known set with cardinality $\mathbf{C}$. The question is, "What known set is most appropriate?" Suppose we try either one of the two sets already in the act. So as not to confuse things, let us introduce a different letter for the individual elements, say w. Let w then also stand for any real number such that

$$0 < w \leq 1.$$

The trick, then, is to form a 1–1 correspondence between elements (x, y) and elements w. In short, we shall try to make each *pair* of nonterminating decimals correspond to a *single* nonterminating decimal, and vice versa.

Actually the task is not too difficult. In all probability you may have already thought of a way. Suppose we select a pair (x_1, y_1) at random with

$$x_1 = .a_1a_2a_3 \cdots$$

and

$$y_1 = .b_1b_2b_3 \cdots$$

where the previous convention involving repeated 9's is adhered to so as to guarantee uniqueness.

Now let

$$w_1 = .a_1b_1a_2b_2a_3b_3 \cdots.$$

This is a nonterminating decimal. It belongs in the appropriate set. To the question, "Is the correspondence actually 1–1?" we suggest the following counter questions: "Given any pair (x, y), can we always determine a unique number w? Given a number w, can we always find a unique pair (x, y)?"

The first is answered already. For the second, let's consider an example. Let $w = .350271643 \cdots$, the dots implying any old continuing sequence of digits. It should be clear that our pattern gives us

$$x = .30763 \cdots$$

and

$$y = .5214 \cdots.$$

Omitting a few final details, we see that the conclusion about

$$\mathbf{C} \cdot \mathbf{C}$$

is inevitable.

We shall now proceed to uncover a major consequence of this result. This is the promised "second dividend." Recall that the set of all real numbers has cardinality $\mathbf{C}$. Since we now know that

$$\mathbf{C} \cdot \mathbf{C} = \mathbf{C},$$

what can be said about the set of all *pairs* of real numbers?

Those having only the barest nodding acquaintance with analytic geometry will recognize that this set of pairs corresponds to the set of all points in the so-called Euclidean plane. Our by-product (no mean achievement!) has thus been the establishment of a rather startling fact, namely, that the set of all points on a line is *equivalent* to the set of all points in a plane.

There is, possibly, an even more surprising result stemming from the same conclusion. One can now assert that:

The set of all points in the plane has the same cardinality as the set of points in the interval $[0, 1]$.

Returning now to an earlier set of questions concerning the arithmetic of transfinite numbers, we may summarize the results obtained thus far as follows:

Assuming that n is any *finite* cardinal number, we have

$$\aleph_0 + n = \aleph_0 + \aleph_0 = \aleph_0$$
$$\aleph_0 \cdot n = \aleph_0 \cdot \aleph_0 = \aleph_0$$

and

$$\mathbf{C} + n = \mathbf{C} + \aleph_0 = \mathbf{C} + \mathbf{C} = \mathbf{C}$$
$$\mathbf{C} \cdot n = \mathbf{C} \cdot \aleph_0 = \mathbf{C} \cdot \mathbf{C} = \mathbf{C}.$$

In terms of order, i.e., comparative size, we know that

$$n < \aleph_0 < \mathbf{C}.$$

8. IS C THE GREATEST?

The time is now ripe to paraphrase the fabled query:

> "Mirror, Mirror, on the wall,
> What is the largest cardinal number of all?"

Is it **C**? If not, how in the world can we find a larger one? Evidently it can't be done by multiplying **C** (the largest thing we've discovered so far) by itself. Furthermore, by applying the associative law we could readily conclude that

$$(\mathbf{C} \cdot \mathbf{C}) \cdot \mathbf{C} = \mathbf{C}$$

and that the same would be true for any finite number of factors. If, in short, we multiply **C** by itself a "zillion" times, we shall obtain no more than **C**. We appear, in other words, to be confronted with the same brand of frustration we met before in respect to $\aleph_0$. That is, we can't seem to enlarge a set by reiterated multiplications or additions.

To find a larger set than one with cardinal number $\aleph_0$, you recall, we were obliged to examine a set we already knew something about, namely, a set of real numbers. It was then possible to show that a 1–1 correspondence could *not* be established between this and the set of positive integers.

It seems reasonable to expect, therefore, that we can now pluck another set out of the blue as it were, and show that this is larger than a set with cardinality **C**. Though such a feat is possible, it seems preferable, at this point, to adopt an alternative procedure, that is, to "create" a new and larger set by means of one we already have. This will, indeed, require some doing!

Let's begin by looking at a phase of transfinite arithmetic which we have already alluded to but not as yet come to actual grips with.

9. CARDINAL EXPONENTS

Given any cardinal number m and another cardinal number n, what shall we mean by the cardinal number

$$m^n?$$

Recall that we are bound by an agreement to preserve the conventional meaning of such a symbol with regard to positive integers when dealing with finite cardinal numbers. We would want our definition to be such that for two finite cardinal numbers, say 2 and 3 for example, the symbol

$$2^3$$

would indicate the cardinal number 8. In addition to this we shall want a meaning for such symbols as

$$2^{\aleph_0},\ \aleph_0^2,\ \aleph_0^{\aleph_0},\ \mathbf{C}^{\aleph_0},\ \aleph_0^{\mathbf{C}},\ \mathbf{C}^{\mathbf{C}}$$

and, in general, a^b where a and b are *any* cardinal numbers whatsoever.

Before we can formulate such a definition we shall need to examine a basic mathematical concept of considerable importance. In all likelihood it is a familiar one. The idea referred to is the general notion of a *function* on one set with values in another. Upon examination it will be discovered that a 1–1 correspondence, about which we have already had so much to say, is an example of such a function. It is a special case, however.

Let us consider, then, a set S and a set T.

Definition: *By a function on S with values in T we shall mean some rule of association which matches each element of S with exactly one element of T.*[1]

It is quite likely that many of you have encountered several alternative definitions in your introduction to the study of functions. The general concept of function is not only basic to the entire realm of mathematics, it is also a subject of considerable controversy in contemporary text-writing circles. Without wishing to become involved in such a controversy, we shall stick to the above definition as

[1] In essence a function involves three entities: a rule of association and two sets.

most appropriate for this particular discussion. This is not intended to restrain the reader from forming his own opinions as he encounters other mathematical literature.

To clarify the notion of a function we offer an illustration. Let S equal the set $\{1, 2, 3\}$ and let T equal the set $\{a, b\}$. Suppose, then, there is a rule which associates 1 with a, 2 with b, and 3 with b.

Symbolically we could write

$$\begin{array}{lcl} 1 \longrightarrow a & & f(1) = a \\ 2 \longrightarrow b & \text{or} & f(2) = b \\ 3 \longrightarrow b & & f(3) = b \end{array}$$

Clearly this correspondence is not 1–1 since both 2 and 3 correspond to the single element b. We sometimes refer to such a correspondence as many-to-one.

With respect to the above notations, the single-headed arrows (in contrast to the double-headed ones appearing previously) imply that the correspondence is not necessarily one-to-one. The symbolism is intended to convey the notion of a one-way street.

The notation on the right is the more conventional method of representing functional values. As with the arrow device, the expression $f(1) = a$ may be interpreted as meaning that the function f associates the element 1 in S with the element a in T. To represent a set of different functions, one may use numerical subscripts as

$$f_1, \quad f_2, \quad f_3, \quad \text{et cetera}$$

or literal subscripts as

$$f_r, \quad f_s, \quad f_t$$

depending on convenience.

One may also use different letters, such as f, g, h, and so forth. All of these representations will appear in the sequel.

Though our definition of function requires that there be an assignment for every element of S, it is not necessary that all of T be included.[1] A function of the type indicated may match all three elements of S with a single element of T. Thus another function is

$$1 \longrightarrow a \qquad 2 \longrightarrow a \qquad 3 \longrightarrow a$$

[1] When all of T is included, such a function is sometimes referred to as a mapping of S *onto* T. When all of T is not included, it is a mapping *into* T. These niceties of distinction needn't trouble us in the present study!

Still another function is

$$\begin{aligned} f_2(1) &= b \\ f_2(2) &= b \\ f_2(3) &= b \end{aligned}$$

Also

$$\begin{aligned} f_3(1) &= b \\ f_3(2) &= b \\ f_3(3) &= a \end{aligned}$$

and so forth.

We now ask the important question, "How many different functions are there for our particular sets?"

For notational convenience we could condense the list as follows:

1	$\longrightarrow a$	$\longrightarrow a$	$\longrightarrow a$	$\longrightarrow b$	$\longrightarrow a$	$\longrightarrow b$	$\longrightarrow b$	$\longrightarrow b$
2	$\longrightarrow a$	$\longrightarrow a$	$\longrightarrow b$	$\longrightarrow a$	$\longrightarrow b$	$\longrightarrow a$	$\longrightarrow b$	$\longrightarrow b$
3	$\longrightarrow a$	$\longrightarrow b$	$\longrightarrow a$	$\longrightarrow a$	$\longrightarrow b$	$\longrightarrow b$	$\longrightarrow a$	$\longrightarrow b$

From the above one can deduce that the number of possible functions is 8, which coincidentally is 2^3.

Though this single example would by no means furnish exhaustive evidence, it does suggest a plausible definition for exponentiation. Before making an explicit statement, however, let's consider a second example. Take the sets K and L where

$$K = \{a, b\} \qquad \text{and} \qquad L = \{x, y, z\}.$$

Suppose we now wish to examine all the possible functions on K with values in L. (We may use the abbreviated expression, "functions *on K to L.*") A few of these are:

$$\begin{array}{lll} f_1(a) = x & f_2(a) = y & f_3(a) = x \\ f_1(b) = y & f_2(b) = x & f_3(b) = z \end{array}$$

As an exercise, complete the list! How many functions are there? If you haven't left any out, the total should be 9. Note also that

$$9 = 3^2.$$

For supportive ammunition let

$$L = \{a, b, c, d\} \quad \text{and} \quad M = \{1, 2, 3\}.$$

Determine the number of possible functions *on L to M*. It may be a little tedious to list them all, but you can, by applying some combinatorial know-how, come up with the answer 81.

Now reverse the situation. Determine the number of functions *on M to L*. In this case the number should be 64. Here again we observe (with less surprise) that

$$81 = 3^4 \quad \text{and} \quad 64 = 4^3.$$

Two ideas emerge: one, there may be a definite connection between functions and exponentiation; two, it may make a significant difference which set is which.

All of this compellingly suggests the following definition.

Definition: *Let S be a set with cardinal number n and let T be a set with cardinal number m. Then m^n is the cardinal number of the set of all possible functions on S with values in T.*

There are three things to keep in mind: (1) The elements of the new set under consideration are functions; (2) The exponent is the cardinal number of the set on which the function is applied; (3) The base is the cardinal number of the set of possible values which the function assumes. In more technical language, the exponent represents the cardinal number of the so-called *domain* of the function, the base represents the cardinal number of the *range*.

Suppose we now let S be the set $\{a, b, c\}$ and let $T = \{0, 1\}$. Then the set of functions *on S to T* will have cardinal number 2^3. A sampling of these functions is

$$\begin{array}{lll} a \longrightarrow 0 & a \longrightarrow 0 & a \longrightarrow 1 \\ b \longrightarrow 1 & b \longrightarrow 0 & b \longrightarrow 0 \\ c \longrightarrow 0 & c \longrightarrow 1 & c \longrightarrow 1 \end{array}$$

Without creating any confusion one could also list the above functions as triples in the following way:

$$(0, 1, 0) \qquad (0, 0, 1) \qquad (1, 0, 1)$$

Thus we might consider the function set as the set of all triples formed with the numbers 0 and 1. As anticipated, there would be 8 of these.

If the set S, on the other hand, contained the five elements $\{a, b, c, d, e\}$, the function set could be represented as a set of all

possible "quintuplets," one of which might have the form

$$(0, 1, 0, 1, 1)$$

indicating, in this case, that

$$a \longrightarrow 0, \quad b \longrightarrow 1, \quad c \longrightarrow 0, \quad d \longrightarrow 1, \quad e \longrightarrow 1$$

How many of these would there be? An excellent answer is 32, or 2^5.

10. THE CARDINAL NUMBER $2^{\aleph_0}$

As you may or may not have suspected, we are in the process of tooling up for a major thrust. It seems now appropriate to face head on a question as to the meaning of

$$2^{\aleph_0}.$$

By definition, it is the cardinal number of the set of all functions on a countably infinite set with values in a set of two elements. For such sets we might just as well choose the set P of all positive integers and the set T consisting, as above, of the elements 0 and 1.

In the notation we have just been using, one such function could be represented, for example, as

$$(1, 0, 1, 1, 0, 0, 0, 1, 0, 1, \cdots)$$

where the sequence of 0's and 1's, in whatever arrangement they happen to be for this particular function, continues indefinitely. Or, let us say, the parentheses contain a countably infinite collection of 0's and 1's.

Having ascertained the general appearance of our function set, it is easy to establish a 1–1 correspondence between this set and the set of all nonterminating decimals, with digits consisting of only 1's and 0's. In fact, the correspondence can be simply obtained by removing the commas and inserting a period. Thus the above sample would correspond to

$$.1011000101 \cdots,$$

which implies that the set of all possible expressions of this form has the same cardinality as the set of functions on P to $\{0, 1\}$.

But every real number in the interval $[0, 1)$ can be expressed in the base 2 system by means of a different expansion of 0's and 1's. If the reader has had limited or no experience with numerical notation using a base of 2 (called variously the *dyadic* or *binary* system of notation), it is strongly suggested that a bit of research be done in this area.

It is not feasible to elaborate on the subject in this particular study. We shall merely give a few illustrative examples to convey the general idea.

Binary notation uses the number 2 as a base, just as the decimal system uses the number 10. In decimal notation, for example, the symbol 534 stands for $5 \cdot 10^2 + 3 \cdot 10 + 4 \cdot 10^0$. Likewise, the symbol .534 would indicate

$$\frac{5}{10} + \frac{3}{10^2} + \frac{4}{10^3} \quad \text{or} \quad (5 \cdot 10^{-1}) + (3 \cdot 10^{-2}) + (4 \cdot 10^{-3}).$$

In binary notation, the symbol $(111)_2$ stands for

$$1 \cdot 2^2 + 1 \cdot 2 + 1 \cdot 2^0, \quad \text{or} \quad 7.$$

As another example,

$$(1101)_2 = (1 \cdot 2^3) + (1 \cdot 2^2) + (0 \cdot 2) + (1 \cdot 2^0), \quad \text{or} \quad 13.$$

For a positive number less than 1, the binary system uses a point in the same manner as in the decimal system. Thus in binary we have

$$(.11)_2 = \frac{1}{2} + \frac{1}{2^2} \quad \text{or} \quad (1 \cdot 2^{-1}) + (1 \cdot 2^{-2}) \quad \text{(.75 in decimals)}$$

while

$$(.1011)_2 = \frac{1}{2} + \frac{0}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}$$

or

$$(1 \cdot 2^{-1}) + (0 \cdot 2^{-2}) + (1 \cdot 2^{-3}) + (1 \cdot 2^{-4}) \quad \text{(.6875 in decimals).}$$

It is clear that the above two quantities could also be written as

$$(.1100 \cdots)_2 \quad \text{and} \quad (.101100 \cdots)_2$$

with the dots signifying a continuation of 0's.[1]

Since, as indicated before, every real number in the interval [0, 1) has a different binary expansion of this type, it is clear that the set of

[1] Should the reader be plagued with lingering uncertainties re the binary system and insufficient time to pursue the matter, he may have to take some parts of the above development on faith.

all real numbers in [0, 1) can be put in 1–1 correspondence with a subset of the set of all such binary expansions. Since, furthermore, the set [0, 1) has cardinality **C**, we note accordingly that

$$\mathbf{C} \leq 2^{\aleph_0}.$$

On the other hand, each infinite *decimal* expansion of the form .101101 ⋯ represents a different real number in the interval [0, 1). Thus it follows that a set of cardinality $2^{\aleph_0}$ is in 1–1 correspondence with a subset of a set with cardinality **C**. Hence

$$2^{\aleph_0} \leq \mathbf{C}.$$

Therefore, it may be asserted, using the corollary to the Bernstein Theorem (see page 43), that

$$2^{\aleph_0} = \mathbf{C}.$$

11. AN ENLARGEMENT PROCESS

We have indeed arrived at a rather momentous conclusion. It has been, in fact, discovered that a set having cardinality $\aleph_0$ can be, in a sense, "enlarged" through exponentiation. Since we already know that $\aleph_0 < \mathbf{C}$, it is evident that

$$\aleph_0 < 2^{\aleph_0}.$$

In words, we have come upon an arithmetic process for obtaining such an enlargement of the countably infinite cardinal number. The inescapable query now confronts us. Would the same hold for any transfinite cardinal number, including our friend **C**? In brief, is $2^{\mathbf{C}}$ larger than **C** itself? If the answer is "Yes," it should be clear to the reader that we may have set off a kind of chain reaction.

Something is definitely in the wind! It behooves us, therefore, to make the essential verification. We shall attempt to do this not only for **C**, but for *any* transfinite number.

Suppose, then, we let k be any cardinal number whatsoever. Consider the cardinal number

$$2^k.$$

What is this number? By definition (see page 52) it is the cardinal number of the set of all functions on a set, say S, having cardinality k, with values in the set $T = \{0, 1\}$.

We shall begin the job by examining a subset of the set of all these functions. Using functional notation, let us denote by f_s a function on S which matches the single element s with 1 and all other elements with zero. For example, suppose S were to contain the elements p, q, r, s, t, u, v, and perhaps others as well. Then in symbols

$$f_s(p) = 0, \quad f_s(q) = 0, \quad f_s(r) = 0, \quad f_s(s) = 1,$$
$$f_s(t) = 0, \quad f_s(u) = 0, \quad f_s(v) = 0,$$

and so forth. Thus f_s might look something like $(0, 0, 0, 1, 0, 0, 0, \cdots)$.

Another function, f_t, for example, matches the element t with 1 and all others with zero. That is,

$$f_t(p) = 0, \quad f_t(q) = 0, \quad f_t(r) = 0, \quad f_t(s) = 0,$$
$$f_t(t) = 1, \quad f_t(u) = 0, \quad f_t(v) = 0, \quad \text{and so on.}$$

In similar manner we define functions f_p, f_q, f_r, f_u, f_v with a subscript to go with each element in S. It is evident that this set of functions does not constitute all possible functions on S to T. There is, for example, a function which matches at least two elements of S with 1. Hence we are considering a subset of the set of all functions, in fact, a proper one.

But there is an obvious 1–1 correspondence between functions of the type f_s and the set of elements of S. (Match subscript with corresponding element!) Thus we can assert that

$$k \leq 2^k.$$

It remains, then, to show that k is not equal to 2^k, whence it would follow that k is actually less than 2^k. We shall first assume that there *is* a 1–1 correspondence between the set of all possible functions on S to $\{0, 1\}$ and the set S itself. We could designate such a correspondence as follows: Let g_s be the function which corresponds to the element s, let g_t correspond to the element t, and so forth. If our correspondence is to be 1–1, then the set of g's must include all possible functions. We shall try to show that it doesn't!

How might one do this? By thinking up a function which couldn't possibly be any of the g's, but which is still a legitimate function on S to $\{0, 1\}$. Suppose we construct a function h which behaves as follows: For any element s in S, let $h(s) = 0$ if $g_s(s) = 1$ (where g_s is the hypothetical correspondent to s). On the other hand, let $h(s) = 1$ if $g_s(s) = 0$.

In case it isn't quite clear what the function h actually is from the above description, it might help to illustrate in terms of a finite example. Take a set with these elements:

$$\text{Let} \quad S = \{x, y, z\}.$$

Now assume (though the assumption looks a bit absurd in this instance) a 1–1 correspondence between S and the set of all functions on S to $\{0, 1\}$. This implies that there are only three such functions, which we may designate as

$$g_x, g_y, g_z.$$

How, then, do we construct the function h? We first investigate the function g_x as applied to the element x. We know that $g_x(x)$ must either equal 1 or 0. We simply, then, let $h(x)$ have the value which $g_x(x)$ does *not* have. Similarly, for $h(y)$ examine the second function g_y.

If $g_y(y) = 1$, let $h(y) = 0$; if $g_y(y) = 0$, let $h(y) = 1$. Do the same with respect to $h(z)$ and $g_z(z)$. Since we do not presume to know just what the individual g functions may be, we cannot exhibit h explicitly. On the other hand, we do know that no matter what the g functions may be, the h function must be different, in one respect at least, from any of them.

If there is any lingering doubt, let's be more specific.

Let
$$g_x(x) = 1, \quad g_y(x) = 0, \quad g_z(x) = 1$$
$$g_x(y) = 1, \quad g_y(y) = 1, \quad g_z(y) = 0$$
$$g_x(z) = 0, \quad g_y(z) = 1, \quad g_z(z) = 0$$

Now describe the function h.

We hope your answer was

$$h(x) = 0, \quad h(y) = 0, \quad h(z) = 1.$$

There is no (0, 0, 1) function among the g's.

It should now be clear that h would also be different from all the g's no matter what or how large the original set S might be. From which it follows that the g set does not contain all functions, contrary to hypothesis. Since the g set represented any possible 1–1 correspondence, there must not be any such animal.

Wherefore be it said that for any cardinal number k

$$k < 2^k.$$

12. CARDINAL NUMBERS LARGER THAN C

We have come a long journey from the original youngsters' debate on the question of a largest number. But perhaps, in a sense, we have arrived full cycle back to much the same position. Whether it be "Mirror, Mirror, on the wall," or the soul-searching query of a young Archimedes, the question, "What is the largest cardinal number of all?" gets the same comeuppance!

There isn't any largest. For suppose the largest cardinal number is a "killion." What, then, about

$$2^{(\text{killion})}?$$

There is an interesting alternative to the notion of 2^k. We can arrive at this by going back, first, to finite cases. Suppose we consider the set

$$S = \{a, b, c\}$$

and wish to determine all possible subsets. It is an accepted fact that the empty set, $\emptyset$, is considered to be a subset of all sets. For if we examine the definition of a subset, we infer that a set T is a subset of a set S if T does not contain any element which is not in S. Now apply this in respect to $\emptyset$ and S.

We have already accepted the fact that any set is a subset of itself. Hence the collection of all subsets of $\{a, b, c\}$ may be displayed as

$$\emptyset, \quad \{a\}, \quad \{b\}, \quad \{c\}, \quad \{a, b\}, \quad \{a, c\}, \quad \{b, c\}, \quad \{a, b, c\}.$$

There seem to be 8 subsets. Furthermore, $8 = 2^3$. But let us wait before jumping to conclusions. Suppose we look at another, and perhaps more ingenious, way of indicating subsets of a given set, say of three elements.

Let the symbol (011), for example, mean a subset of $\{a, b, c\}$ which omits the first element but includes the other two. Thus $(011) = \{b, c\}$, $(101) = \{a, c\}$, and $(100) = \{a\}$.

How, then, would our complete list of subsets look? What about

$$\begin{array}{cccc} (000), & (100), & (010), & (001) \\ (110), & (101), & (011), & (111) \end{array}$$

Surely we've seen something like this before. It looks surprisingly like the set of all functions on $\{a, b, c\}$ to $\{0, 1\}$.

There is no need to labor the point further. We can now supply the alternative definition of the cardinal number 2^k.

It is also the cardinal number of the set of all possible subsets of a set with cardinality k. Again we've a substitute answer to the question about the "largest" cardinal number "killion." Take a set with a "killion" elements. Then find the total number of subsets. How big is this number? Perhaps we can't quite say, but we do know that it is "obviously" larger than a "killion"!

Many loose threads have been left dangling in this brief introduction to transfinite mathematics. Many questions have doubtless been left unanswered. We have not said anything, for example, about

$$\mathbf{C}^{\aleph_0}.$$

We know that it is larger than $\aleph_0$. Why? Because a set with cardinal number $2^{\aleph_0}$ is larger than $\aleph_0$ and such a set can be put in 1-1 correspondence with a subset of a set with cardinality $\mathbf{C}^{\aleph_0}$. There remains, however, the question of whether or not $\mathbf{C}^{\aleph_0}$ is larger than $\mathbf{C}$ itself. This you can try to answer. And many other questions also. We hope not only that you will try, but also that you will succeed. Good luck!

13. SUMMARY

Since the presentation of this material has been somewhat informal, more in the manner of a narrative than a carefully documented sequence of logical consequences, it might be well to close with a summary outline of what we have accomplished. This you may subsequently use as a handy reference. The summary outline follows herewith.

The nonexistence of a "largest" positive integer implies that the set of positive integers goes on forever. Hence the total "number" of positive integers cannot itself be an integer. It must, therefore, be something else—a *transfinite number*. The question is, "Can any type of mathematics be applied to such numbers?" Furthermore, can there be more than one transfinite number? and if so, how can these numbers be determined, classified, combined, and if possible, compared? The task of dealing with these problems is made possible through the use of a branch of set theory.

First, we define the cardinal number of a set as follows:

Definition: *A set S is said to have cardinal number n if and only if the elements of S can be put in* 1-1 *correspondence with the set of integers* $\{1, 2, 3, \cdots, n\}$, *where n is a positive integer.*

Immediately it is seen that the fundamental concept of a 1–1 correspondence between sets lies at the heart of the enterprise.

From here we proceed to place the notion of infinity on a mathematically workable footing as follows:

Definition: *A set S is finite if and only if its cardinal number is a positive integer. A non-empty set which is not finite is called infinite.*

Here we see that the concept of infinite set is predicated on the impossibility of establishing a certain type of 1–1 correspondence.

The set of positive integers proving to be infinite, we assign this set a *transfinite cardinal number* called $\aleph_0$.

Equivalence of sets is then defined by the assertion that:

Definition: *Two sets are equivalent if and only if their elements can be put in* 1–1 *correspondence with each other.*

It follows that any set which is equivalent to the set of positive integers shall likewise be thought of as having the cardinal number $\aleph_0$. This, then, links together three basic notions:

Equivalence 1–1 Correspondence The same cardinal number

From the establishment of a 1–1 correspondence between the set of all positive integers and the set of *even* positive integers, there follows an alternative definition of infinite set:

Definition: *An infinite set is one which can be put in* 1–1 *correspondence with a proper subset of itself.*

Thus the existence of a 1–1 correspondence between one set, say S, and a proper subset of another set, say T, does not guarantee that S has cardinal number less than that of T, as it does in the finite case. We do know, however, that in such instances the cardinal number of S is *less than or equal to* that of T, i.e., $N(S) \leq N(T)$.

An important theorem on equivalence is the Bernstein Theorem, which implies, in effect, that

$$\text{if } N(S) \leq N(T) \text{ and if } N(T) \leq N(S), \text{ then } N(S) = N(T).$$

Whence we obtain the criterion for comparing cardinal numbers:

$$\text{If } N(S) \leq N(T) \text{ but } N(T) \nleq N(S), \text{ then } N(S) < N(T).$$ [1]

[1] The symbol $\nleq$ reads "is *not* less than or equal to."

On an investigatory basis we arrive at an important conclusion, namely, that there are many seemingly larger sets which also have cardinal number $\aleph_0$. Among these are the set of all integers, the set of all rational numbers, and, what's more, the set of all algebraic numbers. It is further established that

$$\aleph_0 \text{ is the } \textit{smallest} \text{ cardinal number}$$

since every infinite set contains a subset with cardinal number $\aleph_0$. Sets with cardinal number $\aleph_0$ are called *countable.*

There follows the search for a set with cardinal number larger than $\aleph_0$. One such set turns out to be the set of all real numbers in the interval (0, 1], to which we assign the cardinal number $\mathbf{C}$. Thus $\aleph_0 < \mathbf{C}_0$.

Many more sets prove to be equivalent to this one, including the set of all points in any finite interval, or half line, or, for that matter, full line. We also deduce that the set of all points in a plane has the cardinal number $\mathbf{C}$.

Before the quest for cardinal numbers larger than $\mathbf{C}$ can be undertaken, some laws of arithmetic for cardinal numbers must be established. These, while designed to include various transfinite numbers, must not be in conflict with existing computational conventions for positive integers.

Accordingly, we define two operations on cardinal numbers as follows:

Definitions:

(1) *If m and n are any two cardinal numbers, then the sum $m + n$ is the cardinal number of any set which is the union of any two disjoint sets having cardinal numbers m and n respectively.*

(2) *The product $m \cdot n$ is the cardinal number of any set which is the Cartesian product (i.e., set of all ordered pairs with an element from each set) of any two sets having cardinal numbers m and n.*

A key factor in the above definitions is the notion that any sum or product depends not on a particular set or sets but merely on any sets having the requisite cardinal numbers. Thus any representative from a class of equivalent sets may be selected in the determination of a given sum or product.

On the basis of these definitions we may make further observations:

Addition and multiplication are both associative and commutative. Multiplication is distributive over addition.

For finite cardinal numbers the usual properties of positive integers are preserved. However, in the case of transfinite cardinal numbers there is a considerable amount of unconventional behavior. This can be summarized as follows:

If n is a finite cardinal number, then

$$\aleph_0 + n = \aleph_0 \qquad \aleph_0 + \aleph_0 = \aleph_0$$
$$\aleph_0 \cdot n = \aleph_0 \qquad \aleph_0 \cdot \aleph_0 = \aleph_0$$
$$\mathbf{C} + n = \mathbf{C} \qquad \mathbf{C} + \aleph_0 = \mathbf{C} \qquad \mathbf{C} + \mathbf{C} = \mathbf{C}$$
$$\mathbf{C} \cdot n = \mathbf{C} \qquad \mathbf{C} \cdot \aleph_0 = \mathbf{C} \qquad \mathbf{C} \cdot \mathbf{C} = \mathbf{C}$$

By associativity, the sum and product results may be expanded to include any finite number of terms or products. Likewise it is shown that

$$\aleph_0 + \aleph_0 + \cdots = \aleph_0$$

for a countable infinity of terms.

A definition follows for exponentiation which yields some very important results:

Definition: *Let S be a set with cardinal number n and let T be a set with cardinal number m. Then m^n is the cardinal number of the set of all functions on S with values in T.*

On the basis of this definition we arrive at a fundamental relationship, namely

$$2^{\aleph_0} = \mathbf{C}.$$

More significant is the final conclusion that for *any* cardinal number k

$$k < 2^k.$$

This provides us with a sound basis for the ultimate assertion that

there is no largest cardinal number.

It can also be shown that for any set with cardinal number k, the set of all possible subsets has cardinal number 2^k. Thus we are furnished with an alternative way of "enlarging" a cardinal number by means of subsets.

14. EPILOGUE

Mathematicians have long been tussling with a celebrated unsolved problem involving transfinite numbers. We have conclusively

shown that $\aleph_0 < \mathbf{C}$. But nowhere could the assertion be made that $\mathbf{C}$ is the "next" number larger than $\aleph_0$. Is it possible to construct a set whose cardinal number is greater than $\aleph_0$ and less than $\mathbf{C}$? Do not be discouraged if you cannot find an immediate answer. No one else has been able to either!

The epic of transfinite numbers has many more verses. There is a very fruitful and challenging companion piece, namely, the study of the so-called *ordinal* numbers. This we most cordially invite you to explore.

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Answers

Page 17.

Positive Rationals		Positive Integers
$\frac{2}{5}$	$\longleftrightarrow$	16
$\frac{5}{3}$	$\longleftrightarrow$	20
$\frac{2}{7}$	$\longleftrightarrow$	26
$\frac{3}{7}$	$\longleftrightarrow$	30
$\frac{3}{8}$	$\longleftrightarrow$	40
$\frac{5}{7}$	$\longleftrightarrow$	44
$\frac{9}{4}$	$\longleftrightarrow$	50

Page 28. 1. 15 2. 10 3. 14 4. 23 5. 25

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Polynomials with height 4:

x^3, $2x^2$, $x^2 + x$, $x^2 - x$, $x^2 + 1$, $x^2 - 1$, $3x$, $2x + 1$, $2x - 1$, $x + 2$, $x - 2$, 4

Polynomials with height 5:

x^4, $2x^3$, $x^3 + x^2$, $x^3 - x^2$, $x^3 + x$, $x^3 - x$, $x^3 + 1$, $x^3 - 1$, $3x^2$, $2x^2 + x$, $2x^2 - x$, $2x^2 + 1$, $2x^2 - 1$, $x^2 + x + 1$, $x^2 - x + 1$, $x^2 + x - 1$, $x^2 - x - 1$, $x^2 + 2x$, $x^2 - 2x$, $x^2 + 2$, $x^2 - 2$, $4x$, $3x + 1$, $3x - 1$, $2x + 2$, $2x - 2$, $x + 3$, $x - 3$, 5

Additional real roots not previously obtained from $h = 2, 3,$ and 4:

-3, $-\frac{1}{2} - \frac{1}{2}\sqrt{5}$, $-\sqrt{2}$, $-\frac{1}{2}\sqrt{2}$, $\frac{1}{2} - \frac{1}{2}\sqrt{5}$, $-\frac{1}{3}$, $\frac{1}{3}$, $-\frac{1}{2} + \frac{1}{2}\sqrt{5}$, $\frac{1}{2}\sqrt{2}$, $\sqrt{2}$, $\frac{1}{2} + \frac{1}{2}\sqrt{5}$, 3

Page 39. 1. $(\frac{1}{16}, \frac{1}{8}]$ of J, $[\frac{1}{16}, \frac{1}{8})$ of K; $(\frac{1}{32}, \frac{1}{16}]$ of J, $[\frac{1}{32}, \frac{1}{16})$ of K

2. $k = \frac{3}{16} - j$; $k = \frac{3}{32} - j$